Cooperation in Networks and Scheduling

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Sebastiaan van Velzen

geboren op 21 augustus 1979 te Naarden.

PROMOTOR:	prof.dr. P.E.M. Borm
COPROMOTORES:	dr. H.J.M. Hamers dr. H.W. Norde

"Vanaf de schepping van de wereld zijn de geleerden aan het denken en toch hebben ze nog niets kunnen bedenken dat even intelligent is als een zoute augurk. "

Uit "Ivanov" van A. Tsjechov (vert. Robert Steijn).

Preface

This thesis is the result of my four years as a PhD-student. I would like to take this opportunity to thank the people whose contribution was indispensable for the realisation of this work.

First of all I would like to thank my two supervisors, Henk and Herbert, for convincing me to become a PhD-student, for all the pleasant meetings we had, and for patiently struggling through all of the unreadable drafts of papers I produced. I also would like to express my gratitude towards my promotor, Peter Borm, and the other members of my thesis committee, Daniel Granot, Mario Bilbao, Tamás Solymosi, Marco Slikker and Stef Tijs, for all the time and effort they spent on evaluating my manuscript. Finally, I would like to thank my coauthors, Flip Klijn, Sílvia Miquel, Marieke Quant and Hans Reijnierse, for our enjoyable cooperation.

Tilburg, April 2005
Bas van Velzen

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Chapter 1

Introduction

1.1 Introduction to game theory

Game theory is a mathematical tool to analyse situations of conflict and cooperation. The two major directions within game theory are non-cooperative game theory and cooperative game theory. Non-cooperative game theory deals with situations of conflict, and cooperative game theory with situations of cooperation. This monograph is mainly concerned with cooperative game theory. The word "game" in the clause "cooperative game" is rather unluckily chosen since the agents involved are generally assumed to have no possibilities to undertake any strategic actions. In fact, in cooperative game theory it is often assumed that binding agreements between the agents are made in order to establish full cooperation. The central question in cooperative game theory is therefore not who will cooperate with whom, but how will the profit generated by this cooperation be divided in a "fair" way? Of course, there does not exist a unique interpretation of the word "fair". Hence, in cooperative game theory there exist many solution concepts, each with its own advantages and disadvantages. The fairness of these solution concepts is mostly measured in terms of properties like monotonicity, consistency and additivity.

The best-known model in cooperative game theory is that of transferable utility games, or TU games for short. A TU game consists of a group of agents, and a value for each subgroup of agents. The value of a subgroup of agents is interpreted as the profit this subgroup can obtain by cooperation. Transferable utility refers to the assumption that utility, for example money, can be transferred from one agent to another. In this monograph we treat several aspects of TU games. We study properties of general TU games and we introduce TU games to model the allocation of cost savings and costs in well-known problems from graph theory and operations research. Before we formally introduce TU games, we first illustrate these games and the most prominent solution concepts and properties that feature in this monograph by means of three examples.

Example 1.1.1 Consider a situation with three agents, referred to as agents 1, 2 and 3, each owning one job that needs to be processed on a machine. It takes 1 time unit for the machine to handle the job of the first agent, 3 time units to handle the job of the second agent, and 2 time units to handle the job of the third agent. Each agent incurs a certain cost as long as his job is not processed on the machine. We assume that the costs of the agents are linear in the completion time of their jobs, and that the completion time cost coefficients are given by 1, 4 and 4 for agents 1, 2 and 3, respectively. So if the job of agent 2 is completed after 5 time units, then he incurs a cost of $5 \cdot 4 = 20$. We summarise the information in Table 1.1.

agent	1	2	3
processing time	1	3	2
cost coefficient	1	4	4

Table 1.1: The processing times and cost coefficients.

We assume that the job of agent 1 is initially scheduled to be processed first, followed by the job of agent 2, and finally the job of agent 3. This processing order yields completion times of 1, $1+3=4$ and $1+3+2=6$ for the jobs of agents 1, 2 and 3, respectively. So the total costs of this processing order equal $1 \cdot 1 + 4 \cdot 4 + 4 \cdot 6 = 41$. However, the agents can generate cost savings by agreeing on a different processing order. For instance, the processing order $(2, 1, 3)$ entails cost of only $4 \cdot 3 + 1 \cdot 4 + 4 \cdot 6 = 40$. The processing order with lowest total costs is $(3, 2, 1)$, which yields total costs of $4 \cdot 2 + 4 \cdot 5 + 1 \cdot 6 = 34$. The main question now is how the agents will distribute these cost savings of $41 - 34 = 7$. In Chapter 6 we formally describe how this situation can be modelled as a cooperative game. At this cooperative game the value of a coalition will be equal to the cost savings this coalition can obtain by reordering its jobs. The game of our example is depicted in Table 1.2.

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	0	1	0	4	7

Table 1.2: The corresponding sequencing game.

The interpretation behind $v(\{1, 2\}) = 1$ is that agents 1 and 2 together can generate total cost savings of 1, without any help of agent 3. Similarly, $v(\{1, 3\}) = 0$ means that agents 1 and 3 together are not able to generate any cost savings. Now that we have modelled the situation as a cooperative game, we can start looking for "fair" allocations. For instance, the allocation $(4, 2, 1)$ is not considered particularly fair since agents 2 and 3 together receive a payoff of 3, while they can achieve cost savings of $v(\{2, 3\}) = 4$ on their own. Hence, $(4, 2, 1)$ is not "stable" in the sense that it gives agents 2 and 3 an incentive not to cooperate with agent 1. This notion of stability is the main idea behind a solution concept known as the core. Intuitively, the core consists of all allocation vectors that distribute the value of the total group of agents such that each subgroup of agents receives an amount that exceeds the value it can achieve on its own. We remark that $(0, 7, 0)$ is a core element of this game, since each subgroup of agents receives at least as much as its stand-alone value. We remark, although the core of this particular game is non-empty, that cores of games can be empty. Since the core is the most prominent solution concept in cooperative game theory, many research is executed to establish sufficient conditions for non-emptiness of the core.

One such sufficient condition is convexity. If a game is convex, then each extreme point of the core coincides with a marginal vector, and the Shapley value is a core element. Convexity is related to marginal contributions of agents to coalitions. For instance, if agent 1 joins the coalition consisting solely of agent 2, then his marginal contribution to this coalition is $v(\{1,2\})-v(\{2\}) = 1$. If he joins coalition $\{2,3\}$, then his marginal contribution is $v(\{1,2,3\})-v(\{2,3\}) = 3$. Here it is the case that the marginal contribution of agent 1 to coalition $\{2,3\}$ exceeds his marginal contribution to the smaller coalition $\{2\}$. A game is called convex if the marginal contribution of any agent to any coalition is larger than his marginal contribution to any smaller coalition. Also related to marginal contributions are marginal vectors. A marginal vector is an allocation vector associated with an order on the player set. An order on the player set is interpreted as the order in which the agents agree to cooperate. The marginal vector associated to this order now allocates to each player precisely his marginal contribution to the coalition he joins. Consider for instance the order $(2,3,1)$. Then agent 2 is the first who agrees to cooperate, followed by agent 3 and finally agent 1. The marginal contributions of the players at this order are $v(\{2\}) - v(\emptyset) = 0$, $v(\{2,3\}) - v(\{2\}) = 4$ and $v(\{1,2,3\}) - v(\{2,3\}) = 3$, for agents 2, 3 and 1, respectively. So the corresponding marginal vector is given by $(3,0,4)$. Note that this marginal vector is a core element.

The Shapley value, probably the best-known one-point solution concept in cooperative game theory, is the average over all marginal vectors. In this example the Shapley value is equal to $(1\frac{1}{6}, 3\frac{1}{6}, 2\frac{2}{3})$. Of course there exist many other solution concepts in cooperative game theory. To give some idea, we just mention the nucleolus and the τ-value. The nucleolus tries to balance the happiness of coalitions, and the τ-value is an average between two vectors expressing minimum and maximum rights of agents. For this example the nucleolus equals $(1\frac{1}{2}, 2\frac{3}{4}, 2\frac{3}{4})$ and the τ-value $(1\frac{5}{16}, 3\frac{1}{16}, 2\frac{5}{8})$. $\diamond$

The following example treats two properties, namely core stability and largeness of the core. Furthermore we discuss tree-component additive games. Tree-component additive games are TU games with a restricted cooperation structure determined by an underlying tree. In fact, only coalitions that are connected with respect to the underlying tree are assumed to be able to fully communicate. Therefore only these connected coalitions are assumed to be able to generate added value. Tree-component additive games are formally defined in Chapter 3, but already illustrated in the following example.

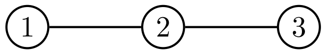

Figure 1.1: A tree (V, E).

Example 1.1.2 Consider the following tree depicted in Figure 1.1. The only disconnected coalition in this tree is $\{1, 3\}$. This means that agents 1 and 3 are not able to communicate and thus not able to generate added value. So if a game is tree-component additive with respect to (V, E), then $v(\{1, 3\}) = v(\{1\}) + v(\{3\})$. The game depicted in Table 1.3 is tree-component additive with respect to (V, E).

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	0	1	0	1	1

Table 1.3: A tree-component additive game.

The vector $(0, 1, 0)$ is the only core element of this game. Indeed, if agent 1 or 3 receives a positive payoff, then coalition $\{2, 3\}$ or coalition $\{1, 2\}$ will be dissatisfied with the payoff it receives. However, the outcome $(0, 1, 0)$ is certainly not the only reasonable outcome of this game. Consider for instance the allocation vector $(\frac{1}{6}, \frac{4}{6}, \frac{1}{6})$. Then both coalitions $\{1, 2\}$ and $\{2, 3\}$ receive a payoff less than their stand-alone values. However, it seems unlikely that one of these coalitions will object to $(\frac{1}{6}, \frac{4}{6}, \frac{1}{6})$ and propose the only core element as the allocation vector, since both player 1 and 3 will not be in favour of this proposal. Such a situation cannot occur in games that satisfy a property named core stability. If a game has a stable core, then for each allocation outside the core, there exists a coalition that will object to this allocation and propose a core element. In general it is difficult to establish whether a game has a stable core. Luckily, there exist sufficient conditions for core stability. One such sufficient condition is largeness of the core. The core of a game is called large if each vector satisfying all coalitions is larger than a core element. The core of the game in our example is not large, because, for instance, the vector $(1, 0, 1)$ is not larger than the only core element, while obviously $(1, 0, 1)$ satisfies all coalitions. $\diamond$

To conclude this section we mention that the values of coalitions do not necessarily have to reflect cost savings, but that these can reflect costs as well. If this is the case, then the cooperative game is called a cost game. Some solution concepts are defined slightly different for cost games, due to the interpretation behind these concepts. We illustrate this in the following example.

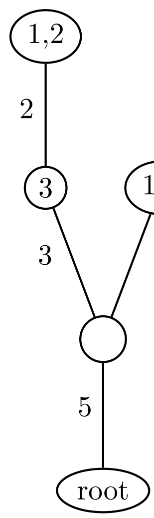

Figure 1.2: A tree depicting a cost sharing problem.

Example 1.1.3 Consider the tree in Figure 1.2, which depicts a cost sharing problem. The numbers in the vertices represent the positions of the agents in the network. Each agent requires at least one connection with the root, and therefore this network needs to be maintained. The numbers next to the edges represent the maintenance cost of these edges. The question is how to divide the total maintenance costs of this network among the agents. In Chapter 5 we formally introduce a cost game modelling this cost allocation problem. The value of a coalition in this game will express the minimum total maintenance costs of this coalition. So for instance the cost of coalition $\{1\}$ is $5+4=9$ and the cost of coalition $\{1,2,3\}$ is $5+3+2=10$. The entire cost game is given by

$$c(S) = \begin{cases} 0, & \text{if } S = \emptyset; \\ 8, & \text{if } S = \{3\}; \\ 9, & \text{if } S = \{1\}, \{4\}, \{1,4\}; \\ 10, & \text{if } S = \{2\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}; \\ 12, & \text{if } S = \{3,4\}, \{1,3,4\}; \\ 14, & \text{if } S = \{2,4\}, \{2,3,4\}, \{1,2,4\}, \{1,2,3,4\}. \end{cases}$$

The core of a cost game consists of all allocation vectors that distribute the value of the total group of agents such that each subgroup of agents is charged an amount less than its stand-alone value. For instance, $(0,8,2,4)$ is a core element of the cost game associated with our example. For cost games, the notion of convexity is replaced by concavity. A cost game is concave if the marginal contribution of any agent to any coalition is smaller than his contribution to a smaller coalition. Concave cost games have non-empty cores. In fact, if a cost game is concave, then each extreme point of the core coincides with a marginal vector and the Shapley value is a core element. Our game is not concave since, for instance, $c(\{1,3,4\})-c(\{1,3\}) = 2 < 4 = c(\{1,2,3,4\}) - c(\{1,2,3\})$. That is, the marginal contribution of agent 4 to coalition $\{1,3\}$ is less than his marginal contribution to coalition $\{1,2,3\}$. $\diamond$

1.2 Games and graphs

In this section we first introduce notation we use throughout this monograph. Then we formally introduce TU games, and several solution concepts and properties. We also recall some basic terminology from graph theory and two duality theorems.

1.2.1 Notation

The set of natural numbers is denoted by $\mathbb{N}$, the set of real numbers by $\mathbb{R}$, the set of non-negative reals by $\mathbb{R}_+$, and the set of positive reals by $\mathbb{R}_{++}$.

For a finite set X, the set $\mathbb{R}^X$ is the space of $|X|$-dimensional vectors with real entries indexed by the elements of X, where $|X|$ denotes the cardinality of X. Throughout this thesis we assume that finite sets are of the form $\{1,\ldots,x\}$, where x is the cardinality of the finite set. Let X be a finite set and let $S \subseteq X$. The vector $e(S) \in \mathbb{R}^X$ is such that $e_i(S) = 1$ if $i \in S$, and 0 otherwise. For any $y \in \mathbb{R}$, y_+ is equal to the maximum of y and 0, i.e. $y_+ = \max\{y, 0\}$, and $\lceil y \rceil$ is the smallest integer exceeding y.

Let N be a finite set. An *order on* N is a bijection from $\{1,\ldots,|N|\}$ to N. If σ is an order on N, then, for each $i \in \{1,\ldots,|N|\}$, $\sigma(i)$ is at the i-th position of σ. An order σ on N will alternatively be denoted by $(\sigma(1),\ldots,\sigma(|N|))$. The set of all orders on N is denoted by $\Pi(N)$. For each $\sigma \in \Pi(N)$, the inverse of σ is denoted by σ^{-1}. So $\sigma^{-1}(i) = j$ if and only if $\sigma(j) = i$. Let $\sigma \in \Pi(N)$, and let $i \in \{1,\ldots,|N|-1\}$. The *i-th neighbour of* σ is the order $\sigma_i \in \Pi(N)$ obtained from σ by interchanging the players at the i-th and $(i+1)$-st position of σ. Formally, $\sigma_i(j) = \sigma(j)$ for each $j \in \{1,\ldots,|N|\}$ with $j \neq i$ and $j \neq i+1$, $\sigma_i(i) = \sigma(i+1)$, and $\sigma_i(i+1) = \sigma(i)$.

The identity order $\sigma^{id} \in \Pi(N)$ is such that $\sigma^{id}(i) = i$ for each $i \in \{1,\ldots,|N|\}$. An order is called *even* if it can be transformed into the identity order by pairwise interchanging the position of players an even number of times. If an order is not even, then it is called *odd*.

1.2.2 Games

A *transferable utility game* (N,v), or TU game for short, consists of a finite player set N and a map $v : 2^N \to \mathbb{R}$. The map $v : 2^N \to \mathbb{R}$ is called the characteristic function and describes for each *coalition* $S \subseteq N$ its *worth* $v(S)$. By convention, $v(\emptyset) = 0$. For convenience, a game (N,v) is sometimes only denoted by v. The set of all TU games with player set N is denoted by TU^N. Let $T \subseteq N$. The subgame associated with coalition T is the game $v_T \in TU^T$, where $v_T(S) = v(S)$ for each $S \subseteq T$.

Let $v \in TU^N$. Coalition $S \subseteq N$ is called *essential* if for each partition P of S it holds that $\sum_{T\in P} v(T) < v(S)$. A coalition which is not essential is *inessential.*

A game $v \in TU^N$ is called *monotone* if for all $S, T \subseteq N$ with $S \subseteq T$,

$$v(S) \leq v(T),$$

and it is called *superadditive* if for each $S, T \subseteq N$ with $S \cap T = \emptyset$,

$$v(S) + v(T) \leq v(S \cup T).$$

A game $v \in TU^N$ is called *convex* if for all $S, T \subseteq N$,

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T). \tag{1.1}$$

Convexity of a game is equivalent to

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T), \tag{1.2}$$

for all $i \in N$ and $S, T \subseteq N \backslash \{i\}$ with $S \subseteq T$. Hence, if a game is convex, then the marginal contribution of a player to a coalition is at most his marginal contribution to a larger coalition. Further, convexity is equivalent to

$$v(S \cup \{i\}) - v(S) \leq v(S \cup \{i, j\}) - v(S \cup \{j\}), \tag{1.3}$$

for all $i, j \in N$, $i \neq j$, and $S \subseteq N \backslash \{i, j\}$, as well.

Let $v \in TU^N$. The *imputation set* $I(v)$ is the set of all efficient and individual rational allocation vectors, i.e.

$$I(v) = \{x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N), x_i \geq v(\{i\}) \text{ for each } i \in N\},$$

and the *core* $C(v)$ is defined by

$$C(v) = \{x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S) \text{ for each } S \subseteq N\}.$$

Intuitively, the core is the set of payoff vectors for which no coalition has an incentive to split off from the grand coalition. We remark that the core of a game can be empty. However, it is shown in Shapley (1971) that convex games have non-empty cores. An important result on non-emptiness of the core is shown in Bondareva (1963) and Shapley (1967). This result uses the definition of so-called balanced collections. A collection $B \subseteq 2^N \backslash \{\emptyset\}$ is called *balanced* if there exists a map $\lambda : B \to (0, 1]$ such that $\sum_{S \in B} \lambda(S) e(S) = e(N)$.

Theorem 1.2.1 (Bondareva (1963), Shapley (1967)) Let $v \in TU^N$. Then $C(v) \neq \emptyset$ if and only if for each balanced collection $B \subseteq 2^N \backslash \{\emptyset\}$ and each map $\lambda : B \to (0,1]$ such that $\sum_{S \in B} \lambda(S) e(S) = e(N)$, it is satisfied that $\sum_{S \in B} \lambda(S) v(S) \leq v(N)$.

The *upper-core* $U(v)$ is the set of not necessarily efficient payoff vectors that are stable against possible split offs, i.e.

$$U(v) = \{x \in \mathbb{R}^N : \sum_{i \in S} x_i \geq v(S) \text{ for each } S \subseteq N\}.$$

Let $v \in TU^N$ and $\sigma \in \Pi(N)$. Then the *marginal vector* $m^\sigma(v)$ associated with σ is defined by

$$m^\sigma_{\sigma(i)}(v) = v([\sigma(i), \sigma]) - v([\sigma(i-1), \sigma]),$$

for each $i \in \{1, \ldots, |N|\}$, where $[\sigma(i), \sigma]$ is the set of predecessors of $\sigma(i)$ with respect to σ. That is, $[\sigma(i), \sigma] = \{\sigma(1), \ldots, \sigma(i)\}$. We will slightly abuse notation by defining $[\sigma(0), \sigma] = \emptyset$ for every $\sigma \in \Pi(N)$. So at a marginal vector each player receives his marginal contribution to the coalition he joins. The *Shapley value* $\Phi(v)$ (Shapley (1953)) can be interpreted as the average of the marginal vectors, i.e.

$$\Phi(v) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} m^\sigma(v).$$

1.2.3 Cost games

In some circumstances the value of a coalition at a TU game is not interpreted as its worth, but as its cost. In those cases we speak of cooperative *cost games* and we denote the game by (N, c). For a cost game $c \in TU^N$, the *core* is given by

$$C(c) = \{x \in \mathbb{R}^N : \sum_{i \in N} x_i = c(N), \sum_{i \in S} x_i \leq c(S) \text{ for each } S \subseteq N\}.$$

Observe that if $c \in TU^N$ is monotone and has a non-empty core, then for each $x \in C(c)$, $x_i = c(N) - \sum_{j \in N \backslash \{i\}} x_j \geq c(N) - c(N \backslash \{i\}) \geq 0$ for every

$i \in N$. That is, each core element of a monotone game is non-negative. A cost game $c \in TU^N$ is called *concave* if for all $S, T \subseteq N$,

$$c(S) + c(T) \geq c(S \cup T) + c(S \cap T).$$

Equivalently, a cost game is concave if and only if for all $i, j \in N$, $i \neq j$, and $S \subseteq N \backslash \{i, j\}$,

$$c(S \cup \{i\}) - c(S) \geq c(S \cup \{i, j\}) - c(S \cup \{j\}).$$

Hence, for concave cost games the marginal contribution of a player to any coalition is at most his marginal contribution to a smaller coalition. We remark that cores of concave cost games are non-empty.

1.2.4 Graphs

A *graph* G is a pair (V, E) where V is a finite set of *vertices*, and E is the set of *edges*, i.e. a set of unordered pairs of V. If $\{v, w\} \in E$ for all distinct $v, w \in V$, then G is called *complete*. The *subgraph induced by* $V' \subseteq V$ is the graph $G_{V'} = (V', E_{V'})$, where $E_{V'}$ is the set of edges having both endpoints in V'.

Two distinct vertices $v, w \in V$ are called *adjacent* if $\{v, w\} \in E$. For $v, w \in V$, a (v, w)-*path* of length m is a sequence $(v, v_1, \ldots, v_{m-1}, w)$ of pairwise distinct vertices, where each subsequent pair of vertices is adjacent, i.e. $\{v, v_1\} \in E$, $\{v_i, v_{i+1}\} \in E$ for all $i \in \{1, \ldots, m-2\}$ and $\{v_{m-1}, w\} \in E$. A *cycle* is a sequence $(v, v_1, \ldots, v_m, v)$ vertices with $m \geq 2$, such that the vertices of $v, v_1, \ldots, v_m$ are pairwise distinct. A graph is said to be *connected* if for any two vertices $v, w \in V$ the graph contains a (v, w)-path. The maximal connected parts of a graph are called *components*.

A *tree* is a connected graph without any cycles. A *leaf* of a tree is a vertex adjacent to only one other vertex. A *chain* is a tree with only two leaves.

1.2.5 Duality theorems

In this section we recall two well-known duality theorems from linear programming. In both theorems we implicitly assume that the feasible regions are non-empty.

Theorem 1.2.2 Let A be an $n \times m$-matrix, $w \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$. Then $\max\{wx : Ax \leq b, x \geq 0\} = \min\{by : yA \geq w, y \geq 0\}$.

Theorem 1.2.3 Let A be an $n \times m$-matrix, $w \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$. Then $\max\{wx : Ax = b, x \geq 0\} = \min\{by : yA \geq w\}$.

1.3 Overview

In this section we give an overview of the contents of this monograph.

In Chapter 2 we study convexity, permutational convexity and marginal vectors. First we focus on the relation between convex games and marginal vectors. Our results strengthen well-known results of Shapley (1971) and Ichiishi (1981), and Rafels and Ybern (1995). Subsequently we study permutational convexity (Granot and Huberman (1982)). We show that permutational convexity is equivalent to a restricted set of inequalities and we introduce a refinement of permutational convexity that is still sufficient for a corresponding marginal vector to be a core element.

Chapter 3 discusses core stability of tree-component additive games and several related concepts such as exactness, largeness and extendibility. A tree-component additive game is a superadditive game with a restricted cooperation structure. Other models in game theory with restricted cooperation possibilities include games with coalition structure (Aumann and Maschler (1964)), partitioning games (Kaneko and Wooders (1982)) and graph-restricted games (Myerson (1977)). Tree-component additive games have also been studied in LeBreton, Owen, and Weber (1991) where non-emptiness of the core is shown, and in Potters and Reijnierse (1995) where it is proved that the core coincides with the bargaining set, and that the kernel consists of the nucleolus only. Solymosi, Aarts, and Driessen (1998) and Kuipers, Solymosi, and Aarts (2000) present algorithms to compute the nucleolus of tree-component additive games.

In Chapter 4 we introduce cooperative games arising from dominating set problems. The dominating set problem is a graph theoretical model that is essentially a location problem. Suppose there is a number of regions that require the service of some facility. Placing a facility in each region is too expensive, and therefore the regions decide to select a subset of the regions, and only place facilities in the selected regions. However, the regions which are not selected need to be served by a facility in a selected region, and so these regions demand that at least one region in their neighbourhood is selected. The first problem the regions face is to select a subset of regions such that the total placement costs are minimised, and all proximity constraints of the regions are met. A second problem is how to divide the total cost among all participating regions. We introduce three cost games to model this cost allocation problem. We focus on the structure and non-emptiness of the core, and we consider concavity as well. Other game theoretical approaches to location problems include facility location games (Kolen and Tamir (1990), Tamir (1992)) and minimum spanning forest games (Granot and Granot (1992)).

In Chapter 5 we discuss a variant of fixed tree games. In a fixed tree problem there is a rooted tree and a group of agents, each agent being located at precisely one vertex of the tree and each vertex containing precisely one agent. The maintenance of each edge in the tree entails a certain cost. The main question is how to assign the total maintenance cost of the tree to the agents. The fixed tree problem was first modelled as a cooperative cost game by Megiddo (1978). Fixed tree games have also been studied in Galil (1980), Granot, Maschler, Owen, and Zhu (1996), Koster, Molina, Sprumont, and Tijs (2001) and Maschler, Potters, and Reijnierse (1995). Variants of fixed tree games where it is allowed that one vertex is occupied by more agents or by no agent are considered in, for example, Koster (1999) and Van Gellekom (2000). However, these variants still require that every agent is located in precisely one vertex. In Chapter 5 we introduce fixed tree problems where agents can occupy more than one vertex. We show that the associated games have non-empty cores, and we study several one-point solution concepts.

Chapter 6 is dedicated to sequencing games. In operations research, a sequencing situation consists of a finite number of jobs and one or more machines. A single decision maker wants to determine a processing schedule of the jobs on the machines that minimises a certain cost criterion, possibly taking into account restrictions like due dates and release times. This single decision maker problem can be turned into a multi decision maker problem by associating an agent to each job. This approach was first taken in Curiel, Pederzoli, and Tijs (1989). They consider sequencing situations with one machine, a linear cost criterion, and no extra restrictions on the jobs. With these sequencing situations they associate the class of sequencing games. Nowadays there exists a wide variety of sequencing games. For instance, Van den Nouweland, Krabbenborg, and Potters (1992), Hamers, Klijn, and Suijs (1999) and Calleja, Borm, Hamers, Klijn, and Slikker (2002) investigate sequencing games arising from multiple-machine sequencing situations. Hamers, Borm, and Tijs (1995) and Borm, Fiestras-Janeiro, Hamers, Sánchez, and Voorneveld (2002) impose release times and due dates on the jobs, respectively. Slikker (2003) considers sequencing games where coalitions are allowed more possibilities to generate cost savings. In Chapter 6 we discuss several sequencing games. First we introduce sequencing games with controllable processing times. These games arise from situations where the processing times are not fixed, but can be reduced at extra cost. We prove non-emptiness of the core in two ways, and we study convexity for some special instances. Subsequently we introduce the class of precedence sequencing games. These games arise from sequencing situations with precedence constraints on the jobs. We show convexity in case the precedence constraints consist of chains and the initial order is a concatenation of these chains. Then we focus on weak-relaxed sequencing games. These games allow some coalitions extra possibilities to generate cost savings. We show non-emptiness of the core by means of permutational convexity. Finally, we consider cooperation in queue allocation of indivisible objects, and show that there exist side-payments that guarantee stability.

Chapter 2

Marginal vectors

2.1 Introduction

Marginal vectors are allocation vectors that divide the worth of the grand coalition using an order on the player set. In particular, each player receives his marginal contribution to the coalition he joins. In literature, several papers explore the relation between marginal vectors and convexity. It is shown in Shapley (1971) that if a game is convex, then all marginal vectors are core elements. The reverse of this statement is shown in Ichiishi (1981). A similar result is proved in Rafels and Ybern (1995). That paper showed that if all even, or all odd marginal vectors are core elements, then the corresponding game is convex.

Granot and Huberman (1982) also studies the relation between convexity and marginal vectors. That paper introduces permutational convexity as a refinement of convexity and show that if an order is permutationally convex for a game, then the marginal vector associated with this order is a core element. By applying this result to minimum cost spanning tree games, it is shown that specific marginal vectors are core elements.

The approach of Granot and Huberman (1982) is adopted by, e.g., Alidaee (1994) and Meca, Timmer, Garcia-Jurado, and Borm (2004). In Alidaee (1994) the permutational convexity of minimum cost spanning forest games is shown, and in Meca, Timmer, Garcia-Jurado, and Borm (2004) the non-emptiness of the core of holding cost games is shown with the use of permutational concavity. In Sections 6.3 and 6.5 of this thesis we will apply permutational convexity to two types of sequencing games.

In this chapter, which is based on Van Velzen, Hamers, and Norde (2002, 2004, 2005), we study marginal vectors, and in particular their relation to convexity. First we show that if two consecutive neighbours of a marginal vector are core elements, then this marginal vector is a core element as well. This result is then used to provide sets of orders that characterise convexity, i.e. a set of orders with the property that the corresponding marginal vectors are only core elements if the corresponding game is convex. In particular, we provide an alternative proof for the result of Rafels and Ybern (1995). Furthermore we investigate the number of orders in minimal convexity characterising sets. Subsequently, we characterise the convexity characterising sets of orders, and we provide a formula for the minimum cardinality of these sets. Finally, we investigate permutational convexity. We introduce a refinement of permutational convexity and show that this refinement is still sufficient for the corresponding marginal vector to be a core element. Furthermore we show that permutational convexity can alternatively be described by a restricted set of inequalities. We conclude the chapter by considering neighbours of permutationally convex orders and we show that if an order is permutationally convex, then its last neighbour is permutationally convex as well.

The remainder of this chapter is organised as follows. In Section 2.2 we recall some early results. In Section 2.3 we take a first approach to finding sets of orders that characterise convexity. In Section 2.4 we provide the formula for the minimum cardinality of convexity characterising sets. Finally, Section 2.5 considers permutational convexity.

2.2 Marginal vectors and convexity

In this section we recall well-known theorems from literature and we introduce permutational convexity.

In Shapley (1971) an important relation between convexity of TU games and marginal vectors is discovered. It is shown that if a game is convex, then all marginal vectors are core elements. In Ichiishi (1981) the reverse is shown. That is, if all marginal vectors belong to the core, then the game is convex. These two results are summarised in the following theorem.

Theorem 2.2.1 (Shapley (1971), Ichiishi (1981)) A game $v \in TU^N$ is convex, if and only if $m^\sigma(v) \in C(v)$ for each $\sigma \in \Pi(N)$.

Theorem 2.2.1 was strengthened in Rafels and Ybern (1995). It is proved in that paper that if all even marginal vectors are core elements, then all odd marginal vectors are core elements as well, and vice versa. Hence, a characterisation of convexity of games is provided by means of $\frac{|N|!}{2}$ specific marginal vectors.

Theorem 2.2.2 (Rafels and Ybern (1995)) Let $v \in TU^N$. Then the following statements are equivalent:

1. (N, v) is convex;
2. $m^\sigma(v) \in C(v)$ for each even $\sigma \in \Pi(N)$;
3. $m^\sigma(v) \in C(v)$ for each odd $\sigma \in \Pi(N)$.

The relation between convexity and marginal vectors is further explored in Granot and Huberman (1982). In that paper permutational convexity is introduced as a refinement of convexity and it is shown that if a game is permutationally convex with respect to an order, then the corresponding marginal vector is a core element of that game.

Let $v \in TU^N$ and $\sigma \in \Pi(N)$. Then (N, v) is *permutationally convex with respect to* σ if

$$v([\sigma(i), \sigma] \cup S) + v([\sigma(k), \sigma]) \leq v([\sigma(k), \sigma] \cup S) + v([\sigma(i), \sigma]) \qquad (2.1)$$

for all $i, k \in \{0, \ldots, |N| - 1\}$ with $i < k$, and $S \subseteq N \backslash [\sigma(k), \sigma]$ with $S \neq \emptyset$. If $v \in TU^N$ is permutationally convex with respect to $\sigma \in \Pi(N)$, then σ is called *permutationally convex for* (N, v).

Theorem 2.2.3 (Granot and Huberman (1982)) Let $v \in TU^N$. If $\sigma \in \Pi(N)$ is permutationally convex for (N, v), then $m^\sigma(v) \in C(v)$.

We remark that the reverse of Theorem 2.2.3 is not true in general. Let $v \in TU^N$. Then checking if $\sigma \in \Pi(N)$ is permutationally convex for (N, v) requires the checking of many inequalities. In fact, for each $i, k \in \{0, \ldots, |N|-1\}$ with $i < k$, there are precisely $2^{|N|-k} - 1$ choices of $S \subseteq N \backslash [\sigma(k), \sigma]$ with $S \neq \emptyset$. Hence, for each $i, k \in \{0, \ldots, |N|-1\}$ with $i < k$, there are precisely $2^{|N|-k} - 1$ permutational convexity inequalities. In total there are

$$\begin{aligned}
\sum_{i=0}^{|N|-2} \sum_{k=i+1}^{|N|-1} [2^{|N|-k} - 1] &= \sum_{i=0}^{|N|-2} [2^{|N|-i} - 2 - (|N| - i - 1)] \\
&= 2^{|N|+1} - 4 - 2(|N|-1) - \frac{1}{2}(|N|-1)|N| \\
&= 2^{|N|+1} - 2 - \frac{1}{2}|N|^2 - 1\frac{1}{2}|N|
\end{aligned}$$

inequalities.

2.3 Neighbour-complete sets

In this section we first show that if two consecutive neighbours of a marginal vector are core elements, then that marginal vector is a core element as well. We will exploit this result to formulate an alternative proof of Theorem 2.2.2. Furthermore we find other sets of orders that provide a characterisation of convexity and we find upper bounds on the number of orders needed to characterise convexity.

We first show that if two consecutive neighbours of a marginal vector are core elements, then that marginal vector is a core element as well.

Lemma 2.3.1 Let $v \in TU^N$ with $|N| \geq 3$ and let $\sigma \in \Pi(N)$. If there is an $h \in \{1, \ldots, |N|-2\}$ with $m^{\sigma_h}(v), m^{\sigma_{h+1}}(v) \in C(v)$, then $m^\sigma(v) \in C(v)$.

Proof: Without loss of generality we assume that $\sigma(i) = i$ for each $i \in \{1, \ldots, |N|\}$. We need to show that $\sum_{i \in S} m_i^\sigma(v) \geq v(S)$ for each $S \subseteq N$.

Let $S \subseteq N$. If $h, h+1 \in S$, or if $h, h+1 \notin S$, then $\sum_{i \in S} m_i^\sigma(v) = \sum_{i \in S} m_i^{\sigma_h}(v) \geq v(S)$. Here the inequality is satisfied because $m^{\sigma_h}(v) \in C(v)$. We now distinguish between two possibilities.

`Case 1:` $h \notin S$, $h+1 \in S$.

Consider coalition $[h, \sigma]$. Then,

$$\begin{aligned}\sum_{i \in [h,\sigma]} m_i^{\sigma_h}(v) &= m_h^{\sigma_h}(v) + \sum_{i \in [h-1,\sigma]} m_i^{\sigma_h}(v) \\ &= v([h+1,\sigma]) - v([h+1,\sigma]\backslash\{h\}) + v([h-1,\sigma]) \\ &\geq v([h,\sigma]).\end{aligned}$$

The inequality is satisfied because $m^{\sigma_h}(v) \in C(v)$. We conclude that

$$v([h+1,\sigma]) - v([h+1,\sigma]\backslash\{h\}) + v([h-1,\sigma]) - v([h,\sigma]) \geq 0. \quad (2.2)$$

Now observe that

$$\begin{aligned}\sum_{i \in S} m_i^{\sigma}(v) &= \sum_{i \in S} m_i^{\sigma_h}(v) - m_{h+1}^{\sigma_h}(v) + m_{h+1}^{\sigma}(v) \\ &= \sum_{i \in S} m_i^{\sigma_h}(v) - \Big[v([h+1,\sigma]\backslash\{h\}) - v([h-1,\sigma])\Big] \\ &\quad + \Big[v([h+1,\sigma]) - v([h,\sigma])\Big] \\ &\geq v(S).\end{aligned}$$

The inequality is satisfied because $m^{\sigma_h}(v) \in C(v)$ implies $\sum_{i \in S} m_i^{\sigma_h}(v) \geq v(S)$, and because of (2.2).

`Case 2:` $h \in S$, $h+1 \notin S$.

If $h+2 \notin S$, then it follows straightforwardly that $\sum_{i \in S} m_i^{\sigma}(v) = \sum_{i \in S} m_i^{\sigma_{h+1}}(v) \geq v(S)$. So assume that $h+2 \in S$. Since now $h+1 \notin S$ and $h+2 \in S$, it can be shown in a similar fashion as in Case 1, that $\sum_{i \in S} m_i^{\sigma}(v) \geq v(S)$. □

Example 2.3.1 Let $v \in TU^N$ with $N = \{1,2,3\}$. The first and second neighbour of $(1,2,3)$ are $(2,1,3)$ and $(1,3,2)$, respectively. From Lemma 2.3.1 it follows that if $m^{(2,1,3)}(v) \in C(v)$ and $m^{(1,3,2)}(v) \in C(v)$, then $m^{(1,2,3)}(v) \in C(v)$. ◇

In Rafels and Ybern (1995) Theorem 2.2.2 is proved by showing that if all even or all odd marginal vectors are core elements, then (1.3) is satisfied for all $i, j \in N$, $i \neq j$, and all $S \subseteq N \backslash \{i, j\}$. We remark that Lemma 2.3.1 provides an alternative proof of Theorem 2.2.2, in case $|N| \geq 3$. Just observe that each neighbour of an even marginal vector is odd, and vice versa. So if all even marginal vectors are core elements, then we can deduce from Lemma 2.3.1 that each odd marginal vector is a core element as well. In fact, Lemma 2.3.1 allows us to obtain different sets of orders that characterise convexity as well. Before we develop such sets, we first introduce some notation.

Let $\{T_1, \ldots, T_k\}$ be a partition of N. Let $\sigma^i \in \Pi(T_i)$ for each $i \in \{1, \ldots, k\}$. Then the combined order $\sigma^1 \cdots \sigma^k \in \Pi(N)$ is that order that begins with the players in T_1 ordered according to σ^1, followed by the players in T_2 ordered according to σ^2, etcetera. The set $\Pi(T_1, \ldots, T_k)$ contains those orders which begin with the players in T_1, followed by the players in T_2, etcetera, i.e. $\Pi(T_1, \ldots, T_k) = \{\sigma^1 \cdots \sigma^k : \sigma^i \in \Pi(T_i) \text{ for every } i \in \{1, \ldots, k\}\}$. These definitions are illustrated in the following example.

Example 2.3.2 Let $N = \{1, 2, 3, 4, 5\}$, $T_1 = \{1, 5\}$, $T_2 = \{3\}$ and $T_3 = \{2, 4\}$. If $\sigma^1 = (5, 1)$, $\sigma^2 = (3)$ and $\sigma^3 = (2, 4)$, then $\sigma^1\sigma^2\sigma^3 = (5, 1, 3, 2, 4)$, and $\Pi(\{1, 5\}, \{3\}, \{2, 4\}) = \{(1, 5, 3, 2, 4), (1, 5, 3, 4, 2), (5, 1, 3, 2, 4),$
$(5, 1, 3, 4, 2)\}$. $\diamond$

Now we introduce an operator $b_T : 2^{\Pi(T)} \to 2^{\Pi(T)}$ for every $T \subseteq N$. For $|T| = 1, 2$, we define $b_T(A) = A$ for all $A \subseteq \Pi(T)$. For $|T| \geq 3$, we define

$$\begin{aligned} b_T(A) \quad = A \ \cup \ \{ \quad & \sigma \in \Pi(T) : \text{ there is an } h \in \{1, \ldots, |T| - 2\} \\ & \text{with } \sigma_h, \sigma_{h+1} \in A\}. \end{aligned}$$

We introduced the operator b_T for the following reason. Let $v \in TU^N$ and $A \subseteq \Pi(N)$. If $m^\sigma(v) \in C(v)$ for each $\sigma \in A$, then it follows from Lemma 2.3.1 that $m^\sigma(v) \in C(v)$ for each $\sigma \in b_N(A)$. So if all marginal vectors corresponding to orders in A are core elements, then all marginal vectors corresponding to orders in $b_N(A)$ are core elements as well. The application of b_N is illustrated in the following example.

Example 2.3.3 Let $N = \{1,2,3,4\}$ and $A = \{(2,1,3,4),(1,3,2,4),(1,4,2,3)\}$. Let $\sigma = (1,2,3,4)$ and $\tau = (1,2,4,3)$. Because $\sigma_1 = (2,1,3,4) \in A$ and $\sigma_2 = (1,3,2,4) \in A$, it follows that $\sigma \in b_N(A)$. Furthermore, note that $\tau \notin A$, $\tau_1 = (2,1,4,3) \notin A$ and $\tau_3 = (1,2,3,4) \notin A$. This implies that $\tau \notin b_N(A)$. However $\tau_2 = (1,4,2,3) \in b_N(A)$ and $\tau_3 = (1,2,3,4) \in b_N(A)$. Therefore, $\tau \in b_N(b_N(A)) = b_N^2(A)$. $\diamond$

In Example 2.3.3 we observed that a repeated application of b_T makes sense. So let $T \subseteq N$. We define the closure of $A \subseteq \Pi(T)$, denoted by $b_T^*(A)$, to be the largest set of orders that can be obtained by repetitive application of b_T, i.e. $b_T^*(A) = b_T^k(A)$ for $k \in \mathbb{N}$ with $b_T^k(A) = b_T^{k+1}(A)$.

Let $C \subseteq \Pi(T)$. If $A \subseteq C$ is such that $C \subseteq b_T^*(A)$, then A is called *neighbour-complete*, or *n-complete*, *in* C. If $A \subseteq \Pi(T)$ is n-complete in $\Pi(T)$, then A is called *n-complete*. From Lemma 2.3.1 it follows that if $A \subseteq \Pi(N)$ is n-complete, then $m^\sigma(v) \in C(v)$ for each $\sigma \in A$ implies that (N,v) is convex. Before we obtain n-complete sets, we first find n-complete sets for $\Pi(T_1,\ldots,T_k)$, where $\{T_1,\ldots,T_k\}$ is a partition of N.

Lemma 2.3.2 Let $\{T_1,\ldots,T_k\}$ be a partition of N and let $A_i \subseteq \Pi(T_i)$. If A_i is n-complete in $\Pi(T_i)$ for each $i \in \{1,\ldots,k\}$, then $A = \{\tau^1\cdots\tau^k : \tau^i \in A_i \text{ for each } i \in \{1,\ldots,k\}\}$ is n-complete in $\Pi(T_1,\ldots,T_k)$.

Proof: Let $\sigma^1\cdots\sigma^k \in \Pi(T_1,\ldots,T_k)$. We use induction on j to show for each $j \in \{1,\ldots,k+1\}$, that $\sigma^1\cdots\sigma^{j-1}\tau^j\cdots\tau^k \in b_N^*(A)$ for all $(\tau^j,\ldots,\tau^k) \in A_j \times \ldots \times A_k$. This shows, with $j = k+1$, that $\sigma^1\cdots\sigma^k \in b_N^*(A)$.

First note, for the induction basis, that $\tau^1\cdots\tau^k \in b_N^*(A)$ for all $(\tau^1,\ldots,\tau^k) \in A_1 \times \ldots \times A_k$.

As the induction hypothesis, assume that $j^* \in \{1,\ldots,k+1\}$ is such that for all $j \in \{1,\ldots,j^*\}$ it is satisfied that $\sigma^1\cdots\sigma^{j-1}\tau^j\cdots\tau^k \in b_N^*(A)$ for all $(\tau^j,\ldots,\tau^k) \in A_j \times \ldots \times A_k$.

If $j^* = k+1$, then we are done, so assume that $j^* < k+1$. Let $(\tau^{j^*+1},\ldots,\tau^k) \in A_{j^*+1} \times \ldots \times A_k$ and let $C(\tau^{j^*+1},\ldots,\tau^k) = \{\sigma^1\cdots\sigma^{j^*-1}\tau\tau^{j^*+1}\cdots\tau^k : \tau \in A_{j^*}\}$. According to our induction hypothesis, $C(\tau^{j^*+1},\ldots,\tau^k) \subseteq b_N^*(A)$. Because $\sigma^{j^*} \in b_{T_{j^*}}^*(A_{j^*})$ it follows that

$\sigma^1 \cdots \sigma^{j^*-1}\sigma^{j^*}\tau^{j^*+1} \cdots \tau^k \in b_N^*(C) \subseteq b_N^*(A)$. Therefore $\sigma^1 \cdots \sigma^{j^*}\tau^{j^*+1} \cdots \tau^k \in b_N^*(A)$ for all $(\tau^{j^*+1}, \ldots, \tau^k) \in A_{j^*+1} \times \ldots \times A_k$. □

Example 2.3.4 Let $N = \{1, \ldots, 6\}$, $T_1 = \{1, 2, 4\}$ and $T_2 = \{3, 5, 6\}$. Let $A_1 = \{(1, 2, 4), (2, 4, 1), (4, 1, 2)\}$ and $A_2 = \{(3, 5, 6), (5, 6, 3), (6, 3, 5)\}$. Note that A_1 and A_2 are n-complete in $\Pi(T_1)$ and $\Pi(T_2)$, respectively. It follows from Lemma 2.3.2 that $A = \{\sigma\tau : \sigma \in A_1, \tau \in A_2\}$ is n-complete in $\Pi(\{1, 2, 4\}, \{3, 5, 6\})$. ◇

In the remainder of this section we focus on the cardinality of n-complete sets. We will call $A \subseteq \Pi(T)$ *minimum n-complete in* $\Pi(T)$ if it is an n-complete set in $\Pi(T)$ of minimum cardinality. Because of symmetry, the cardinality of minimum n-complete sets in $\Pi(T)$ does not depend on T, but only on the cardinality of T. Let the *neighbour number* Q_t denote the cardinality of a minimum n-complete set in $\Pi(T)$, where $t = |T|$.[1] So $Q_t = \min\{|A| : A \subseteq \Pi(T) \text{ is n-complete in } \Pi(T)\}$. By definition, $Q_1 = 1$ and $Q_2 = 2$. From Theorem 2.2.2, and our alternative proof of this theorem, it follows that $Q_n \leq \frac{n!}{2}$ for each $n \geq 3$. Finally, define the *relative neighbour number* $F_n = \frac{Q_n}{n!}$. The following proposition, which is a direct consequence of Lemma 2.3.2, gives a strengthening of the bound obtained from Theorem 2.2.2.

Proposition 2.3.1 Let $n_1, \ldots, n_k, n \in \mathbb{N}$ be such that $\sum_{i=1}^k n_i = n$. Then $Q_n \leq \frac{n!}{n_1! \cdots n_k!} \prod_{i=1}^k Q_{n_i}$.

Proof: Let N be a finite set of cardinality n. Observe that $\Pi(N)$ can be partitioned into $\frac{n!}{n_1! \cdots n_k!}$ sets of the form $\Pi(T_1, \ldots, T_k)$, with $|T_i| = n_i$ for each $i \in \{1, \ldots, k\}$, and $\{T_1, \ldots, T_k\}$ a partition of N. Now let $\{T_1, \ldots, T_k\}$ be a partition of N with $|T_i| = n_i$ for each $i \in \{1, \ldots, k\}$. The n-complete set in $\Pi(T_1, \ldots, T_k)$ from Lemma 2.3.2 contains $\prod_{i=1}^k Q_{n_i}$ elements. Hence, $Q_n \leq \frac{n!}{n_1! \cdots n_k!} \prod_{i=1}^k Q_{n_i}$. □

From Proposition 2.3.1 it follows that $Q_{n+1} \leq \frac{(n+1)!}{n!1!} Q_n Q_1 = (n + 1)Q_n$. This implies that $F_{n+1} \leq F_n$ for each $n \in \mathbb{N}$. So F_n is non-increasing. In

[1] In the remainder of this section we denote the cardinality of a finite set T by t. Similarly, the cardinality of a finite set N is denoted by n.

fact, the next theorem exploits Proposition 2.3.1 to conclude that $F_n \to 0$ if $n \to \infty$. So the relative number of orders needed to characterise convexity converges to zero.

Theorem 2.3.1 If $n \to \infty$, then $F_n \to 0$.

Proof: Let $k \in \mathbb{N}$, and $n_i = 3$ for every $i \in \{1, \ldots, k\}$. From Proposition 2.3.1 we deduce, using $n = 3k$, that

$$Q_{3k} \leq \frac{(3k)!}{(3!)^k} 3^k.$$

Here we have used that $Q_3 \leq 3$. We conclude that $F_{3k} \leq (\frac{1}{2})^k$ for every $k \in \mathbb{N}$ and therefore that $F_{3k} \to 0$ if $k \to \infty$. Because $F_{n+1} \leq F_n$ for each $n \in \mathbb{N}$, it follows that $F_n \to 0$ if $n \to \infty$. $\square$

The final result of this section gives lower bounds for Q_n.

Proposition 2.3.2 If $n \in \mathbb{N}$ is even, then $Q_n \geq n! \frac{1}{2^{\frac{n-2}{2}}}$. If $n \in \mathbb{N}$ is odd, then $Q_n \geq n! \frac{1}{2^{\frac{n-1}{2}}}$.

Proof: Let $n \in \mathbb{N}$ be even, let $N = \{1, \ldots, n\}$, and let $k = \frac{n+2}{2}$. Let $\{T_1, \ldots, T_k\}$ be a partition of N, with $|T_1| = |T_k| = 1$, and $|T_i| = 2$ for each $i \in \{2, \ldots, k-1\}$. Let $C = \Pi(T_1, \ldots, T_k)$. Note that if $\sigma \in C$, then $\sigma_i \in C$ for each even $i \in \{1, \ldots, n-1\}$.

Now let $A \subseteq \Pi(N)$ be such that $A \cap C = \emptyset$. Let $\sigma \in C$. Because for each $i \in \{1, \ldots, n-2\}$ either i or $i+1$ is even, we conclude that for all $i \in \{1, \ldots, n-2\}$ it is satisfied that $\sigma_i \in C$, or $\sigma_{i+1} \in C$. This implies for every $i \in \{1, \ldots, n-2\}$ that $\sigma_i \notin A$ or $\sigma_{i+1} \notin A$. Hence, $\sigma \notin b_N(A)$. We conclude that $b_N(A) \cap C = \emptyset$. By repetition, we find $b_N^*(A) \cap C = \emptyset$.

So if $A \subseteq \Pi(N)$ is n-complete in $\Pi(N)$, then it is true that $|A \cap C| \geq 1$. Since we can partition $\Pi(N)$ into $\frac{n!}{2^{k-2}}$ sets of the form $\Pi(T_1, \ldots, T_k)$, with $\{T_1, \ldots, T_k\}$ a partition of N, $|T_1| = |T_k| = 1$, and $|T_i| = 2$ for every $i \in \{2, \ldots, k-1\}$, we conclude that $Q_n \geq \frac{n!}{2^{k-2}} = n! \frac{1}{2^{\frac{n-2}{2}}}$.

Now let $n \in \mathbb{N}$ be odd, let $N = \{1, \ldots, n\}$, and let $k = \frac{n+1}{2}$. Let $\{T_1, \ldots, T_k\}$ be a partition of N with $|T_1| = 1$, and $|T_i| = 2$ for every

$i \in \{2, \ldots, k\}$. We can conclude, similarly to the case where n is even, that if A is complete, then $A \cap \Pi(T_1, \ldots, T_k) \neq \emptyset$. Because $\Pi(N)$ can be partitioned into $\frac{n!}{2^{k-1}}$ sets of the form $\Pi(T_1, \ldots, T_k)$, with $\{T_1, \ldots, T_k\}$ a partition of N, $|T_1| = 1$, and $|T_i| = 2$ for every $i \in \{2, \ldots, k\}$, we conclude that $Q_n \geq \frac{n!}{2^{k-1}} = n! \frac{1}{2^{\frac{n-1}{2}}}$. □

Now combining Theorem 2.2.2 with Proposition 2.3.2 gives $Q_3 = 3$ and $Q_4 = 12$. Furthermore, we obtain from Theorem 2.2.2 that $Q_5 \leq 60$ and from Proposition 2.3.2 that $Q_5 \geq 30$. Therefore $Q_5 \in [30, 60]$. However, in Van Velzen, Hamers, and Norde (2002) it is shown, using ad hoc methods, that $Q_5 = 30$. Some other bounds are given in Table 2.1. These other bounds all follow from Propositions 2.3.1 and 2.3.2.

n	3	4	5	6	7	8	9
$n!$	6	24	120	720	5040	40320	362880
$\frac{n!}{2}$	3	12	60	360	2520	20160	181440
Q_n	3	12	30	180	[630,1260]	5040	[22680,45360]
F_n	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$[\frac{1}{8}, \frac{1}{4}]$	$\frac{1}{8}$	$[\frac{1}{16}, \frac{1}{8}]$

Table 2.1: New bounds

2.4 A characterisation of minimum cardinality

In this section we continue our exploration of convexity characterising sets. However, in this section we trail a more efficient method than that of Section 2.3. First we introduce complete sets, and then characterise these sets.

A set $A \subseteq \Pi(N)$ is said to be *complete* if for every $v \in TU^N$ the following assertions are equivalent:

1. (N, v) is convex;

2. $m^\sigma(v) \in C(v)$ for each $\sigma \in A$.

First note, according to Theorem 2.2.1, that $\Pi(N)$ is a complete set. From Theorem 2.2.2 we deduce that the sets of even and odd orders are also complete sets. Furthermore we remark that each n-complete set is complete as well. We are interested in the minimum cardinality of complete sets. Therefore we introduce

$$M_n = \min\{|A| : A \subseteq \Pi(N) \text{ is complete}\},$$

where $n = |N|$.[2] Note that $M_n \leq Q_n$ for each $n \in \mathbb{N}$. Before we characterise complete sets, we first introduce some notation. Let $i, j \in N$, $i \neq j$, and $S \subseteq N \backslash \{i, j\}$. Then $P(S, \{i, j\})$ is the set of orders that begin with the players in S, followed by the players in $\{i, j\}$, and end with the players in $N \backslash (S \cup \{i, j\})$. Note that we allow for $S = \emptyset$ and $S = N \backslash \{i, j\}$. We are now ready for our characterisation of complete sets.

Lemma 2.4.1 The set $A \subseteq \Pi(N)$ is complete if and only if

$$A \cap P(S, \{i, j\}) \neq \emptyset \tag{2.3}$$

for all $i, j \in N$, $i \neq j$, and $S \subseteq N \backslash \{i, j\}$.

Proof: First we show the "if" part. Let $v \in TU^N$ and let $A \subseteq \Pi(N)$ satisfy (2.3). We need to show that $m^\sigma(v) \in C(v)$ for each $\sigma \in A$ implies that (N, v) is convex. So assume that $m^\sigma(v) \in C(v)$ for each $\sigma \in A$. For showing that (N, v) is convex, we need to show that (1.3) is satisfied for all $i, j \in N$, $i \neq j$ and all $S \subseteq N \backslash \{i, j\}$. So let $i, j \in N$, $i \neq j$, and $S \subseteq N \backslash \{i, j\}$. By assumption there is a $\sigma \in A$ with $\sigma \in P(S, \{i, j\})$. Assume without loss of generality that $\sigma(|S| + 1) = i$ and $\sigma(|S| + 2) = j$. Then,

$$v(S \cup \{j\}) \leq \sum_{k \in S \cup \{j\}} m_k^\sigma(v) = v(S \cup \{i, j\}) - v(S \cup \{i\}) + v(S).$$

The inequality is satisfied because $m^\sigma(v) \in C(v)$. We conclude that (N, v) is convex.

It remains to show the "only if" part. Assume that $A \subseteq \Pi(N)$ does not satisfy (2.3). We will show that A is not complete by constructing a non-convex game for which all marginal vectors corresponding to orders in A are core elements.

[2] In the remainder of this section we denote the cardinality of a finite set N by n.

Because A does not satisfy (2.3), there are $i,j \in N$, $i \neq j$, and $S \subseteq N\backslash\{i,j\}$ with $A \cap P(S,\{i,j\}) = \emptyset$. Define $v \in TU^N$ by

$$v(T) = \begin{cases} 1 & \text{if } \quad T = S \cup \{i\}, S \cup \{j\} \\ \max(0, |T| - |S| - 1) & \text{otherwise.} \end{cases}$$

We will show that $m^\sigma(v) \in C(v)$ if and only if $\sigma \notin P(S,\{i,j\})$. This implies that (N,v) is not convex, and that $m^\sigma(v) \in C(v)$ for each $\sigma \in A$.

First we show that $m^\sigma(v) \notin C(v)$ for each $\sigma \in P(S,\{i,j\})$. Let $\sigma \in P(S,\{i,j\})$. Without loss of generality assume that $\sigma(|S|+1) = i$ and $\sigma(|S|+2) = j$. Then

$$\begin{aligned} \sum_{k \in S \cup \{j\}} m_k^\sigma(v) &= v(S \cup \{i,j\}) - v(S \cup \{i\}) + v(S) \\ &= 1 - 1 + 0 \\ &< 1 \\ &= v(S \cup \{j\}). \end{aligned}$$

Hence, $m^\sigma(v) \notin C(v)$. It remains to show that $m^\sigma(v) \in C(v)$ for each $\sigma \notin P(S,\{i,j\})$.

Let $\sigma \notin P(S,\{i,j\})$. First observe that by definition of (N,v), it is satisfied that $v(T \cup \{k\}) - v(T) \in \{0,1\}$ for all $k \in N$ and $T \subseteq N\backslash\{k\}$. This implies that

$$m_k^\sigma(v) \in \{0,1\} \tag{2.4}$$

for each $k \in N$. Now let $T \subseteq N$. For showing that $\sum_{k \in T} m_k^\sigma(v) \geq v(T)$ we distinguish between two cases.

`Case 1:` $T \neq S \cup \{i\}, S \cup \{j\}$.

If $|T| \leq |S| + 1$, then it follows by definition of (N,v) that $v(T) = 0$. This implies, using (2.4), that $\sum_{k \in T} m_k^\sigma(v) \geq 0 = v(T)$.

If $|T| > |S| + 1$, then we conclude using (2.4) that $\sum_{k \in N\backslash T} m_k^\sigma(v) \leq$

$|N\backslash T|$. This implies

$$\begin{aligned}\sum_{k\in T} m_k^\sigma(v) &= v(N) - \sum_{k\in N\backslash T} m_k^\sigma(v)\\ &\geq |N| - |S| - 1 - |N\backslash T|\\ &= |T| - |S| - 1\\ &= v(T).\end{aligned}$$

`Case 2:` $T = S\cup\{i\}$ or $T = S\cup\{j\}$.

Without loss of generality assume that $T = S\cup\{i\}$. Observe that $v(S\cup\{i\}) = 1$. Because of (2.4), it is sufficient to prove there is a $k \in S\cup\{i\}$ with $m_k^\sigma(v) = 1$.

Let $h \in S\cup\{i\}$ be that player ordered last with respect to σ, i.e. $\sigma^{-1}(k) \leq \sigma^{-1}(h)$ for all $k \in S\cup\{i\}$. Note that by definition, $\sigma^{-1}(h) \geq |S| + 1$. We distinguish between three subcases.

`Subcase 2a:` $\sigma^{-1}(h) = |S| + 1$.

So σ is an order that starts with the members of $S\cup\{i\}$. Hence, $m_h^\sigma(v) = v(S\cup\{i\}) - v((S\cup\{i\})\backslash\{h\}) = 1 - 0 = 1$.

`Subcase 2b:` $\sigma^{-1}(h) = |S| + 2$.

First assume that $h \neq i$. Then $h \in S$ and this implies that $[h,\sigma]\backslash\{h\} \neq S\cup\{i\}, S\cup\{j\}$. So $v([h,\sigma]) = 1$ and $v([h,\sigma]\backslash\{h\}) = 0$. We conclude that $m_h^\sigma(v) = v([h,\sigma]) - v([h,\sigma]\backslash\{h\}) = 1 - 0 = 1$.

Secondly, assume that $h = i$. If $\sigma^{-1}(j) > \sigma^{-1}(h)$, then $[h,\sigma]\backslash\{h\} \neq S\cup\{j\}$, and because $h = i$ it is satisfied that $[h,\sigma]\backslash\{h\} \neq S\cup\{i\}$. This implies that $v([h,\sigma]\backslash\{h\}) = 0$. From $v([h,\sigma]) = 1$ we conclude that $m_h^\sigma(v) = v([h,\sigma]) - v([h,\sigma]\backslash\{h\}) = 1 - 0 = 1$.

If $\sigma^{-1}(j) < \sigma^{-1}(h)$, then it follows from $\sigma \notin P(S,\{i,j\})$ that $\sigma(|S|+1) \neq j$. Now let $k \in S$ be such that $\sigma(|S|+1) = k$. Then, $[k,\sigma] = S\cup\{j\}$. This implies that $m_k^\sigma(v) = v([k,\sigma]) - v([k,\sigma]\backslash\{k\}) = 1 - 0 = 1$.

`Subcase 2c:` $\sigma^{-1}(h) \geq |S| + 3$.

Then $|[h,\sigma]| \geq |S| + 3$, and $|[h,\sigma]\backslash\{h\}| = |[h,\sigma]| - 1$. Hence, $m_h^\sigma(v) = v([h,\sigma]) - v([h,\sigma]\backslash\{h\}) = (|[h,\sigma]| - |S| - 1) - (|[h,\sigma]\backslash\{h\}| - |S| - 1) = 1$. $\square$

Before we give the formula for the minimum cardinality of complete sets, it is convenient to introduce some terminology. Let N be a finite set and define, for each $k \in \{0, \dots, n-2\}$,

$$G_n(k) = \{P(S, \{i, j\}) : i, j \in N, i \neq j, S \subseteq N \backslash \{i, j\} \text{ and } |S| = k\}$$

as the collection of sets $P(S, \{i, j\})$ where S contains precisely k members. Now let $k \in \{0, \dots, n-2\}$. Obviously, for each $\sigma \in \Pi(N)$ there is precisely one $P(S, \{i, j\}) \in G_n(k)$ with $\sigma \in P(S, \{i, j\})$. In other words, $G_n(k)$ is a partition of $\Pi(N)$ for each $k \in \{0, \dots, n-2\}$. Observe that $|G_n(k)| = \binom{n}{k}\binom{n-k}{2}$.

From Lemma 2.4.1 it follows that complete sets are those sets that cover all elements of $G_n(k)$, for each $k \in \{0, \dots, n-2\}$. That is, $A \subseteq \Pi(N)$ is complete if and only if $A \cap B \neq \emptyset$ for all $B \in \bigcup_{k=0}^{n-2} G_n(k)$. So we can easily find complete sets by choosing, for each $k \in \{0, \dots, n-2\}$, an order from every $B \in G_n(k)$. In this way we obtain a complete set containing at most $\sum_{k=0}^{n-2} |G_n(k)|$ orders. Of course, there are complete sets containing less than $\sum_{k=0}^{n-2} |G_n(k)|$ orders. The main result of this section is the formula for the minimum cardinality of a complete set. In particular, we give a method to construct minimum cardinality complete sets. It turns out to be convenient to distinguish between odd $n \in \mathbb{N}$ and even $n \in \mathbb{N}$. Therefore we distinguish between those two possibilities.

First we focus on odd $n \in \mathbb{N}$. Let $n \in \mathbb{N}$, $n \geq 3$, be odd and let $N = \{1, \dots, n\}$. First we introduce the concepts of right-hand side and left-hand side neighbours, and that of perfect coverings. Let $k = \frac{n-1}{2}$. Assume that the players are seated at a round table such that for all $j \in N$ the person on the right-hand side of player j is player $(j-1)$ *mod* n and the person on his left-hand side is player $(j+1)$ *mod* n. For each $j \in N$, the set of *right-hand side neighbours* of j, denoted by R_j, consists of the first k players on the right-hand side of player j, i.e.

$$R_j = \{(j-1) \textit{ mod } n, \dots, (j-k) \textit{ mod } n\}.$$

Similarly, for each $j \in N$, the set of *left-hand side neighbours* of j, denoted by L_j, consists be the first k players on the left-hand side of player j, i.e.

$$L_j = \{(j+1)\ mod\ n, \ldots, (j+k)\ mod\ n\}.$$

The notion of left-hand side neighbours and right-hand side neighbours is illustrated in Example 2.4.1.

Example 2.4.1 Let $N = \{1, \ldots, 9\}$, $k = 4$ and $j = 3$. Then $R_3 = \{1, 2, 8, 9\}$ and $L_3 = \{4, 5, 6, 7\}$. These sets are illustrated in Figure 2.1. $\diamond$

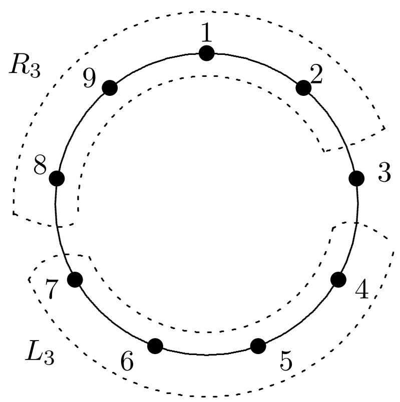

Figure 2.1: The left-hand side and right-hand side neighbours of 3.

It is straightforward to verify the following properties of R_j and L_j, for all $i, j \in N$.

(P1) $L_j \cap R_j = \emptyset$;

(P2) $L_j \cup R_j \cup \{j\} = N$;

(P3) $i \in L_j$ if and only if $j \in R_i$;

(P4) $i \in R_j$ if and only if $j \notin R_i$.

Now we introduce the concept of perfect coverings, which is closely related to the concepts of right-hand side and left-hand side neighbours. Let $i, j \in N$, $i \neq j$, and $S \subseteq N \backslash \{i, j\}$. Then $\sigma \in P(S, \{i, j\})$ is said to *perfectly cover* $P(S, \{i, j\})$ if $\sigma(|S|+1) \in R_{\sigma(|S|+2)}$. It is easily verified, using (P4), that half the number of orders in $P(S, \{i, j\})$ perfectly covers this set. We illustrate perfect coverings in the following example.

Example 2.4.2 Let $N = \{1, \ldots, 9\}$, $S = \{1, 4, 5\}$, $i = 8$ and $j = 3$. Then, $8 \in R_3$, but $3 \notin R_8$. Hence, $\sigma \in P(\{1, 4, 5\}, \{3, 8\})$ is a perfect covering of this set if and only if $\sigma(4) = 8$ and $\sigma(5) = 3$. So, for instance, $(5, 1, 4, 8, 3, 2, 9, 6, 7)$ is a perfect covering of $P(\{1, 4, 5\}, \{3, 8\})$, but $(5, 1, 4, 3, 8, 2, 9, 6, 7)$ is not. $\diamond$

The last concept we introduce before we state the formula of minimum cardinality of complete sets is that of perfect completeness. A set $A \subseteq \Pi(N)$ is called *perfect complete* if for each $i, j \in N$, $i \neq j$, and $S \subseteq N \backslash \{i, j\}$ there is a $\sigma \in A$ that perfectly covers $P(S, \{i, j\})$. Observe that if a set is perfect complete, then it follows by Lemma 2.4.1 that it is complete as well.

The following theorem provides the minimum cardinality of a complete set for odd $n \in \mathbb{N}$. The proof of this theorem is constructive in the sense that it contains a procedure to obtain a perfect complete set of minimum cardinality.

Theorem 2.4.1 Let $n \in \mathbb{N}$ with $n \geq 3$ be odd. Then

$$M_n = \frac{n!}{2(\frac{n-3}{2})!(\frac{n-1}{2})!}.$$

Proof: Let $k = \frac{n-1}{2}$. First we show that $M_n \geq \frac{n!}{2(\frac{n-3}{2})!(\frac{n-1}{2})!}$. Since $G_n(k)$ forms a partition of $\Pi(N)$, it follows that to cover all elements of $G_n(k)$ at least $|G_n(k)|$ orders are needed. Note that $|G_n(k)| = \binom{n}{k}\binom{n-k}{2} = \frac{n!}{k!(n-k-2)!2!} = \frac{n!}{2(\frac{n-3}{2})!(\frac{n-1}{2})!}$. Therefore, using Lemma 2.4.1, it follows that $M_n \geq \frac{n!}{2(\frac{n-3}{2})!(\frac{n-1}{2})!}$.

Now we show that $M_n \leq \frac{n!}{2(\frac{n-3}{2})!(\frac{n-1}{2})!}$. We do this by inductively constructing a perfect complete set of size $|G_n(k)| = \frac{n!}{2(\frac{n-3}{2})!(\frac{n-1}{2})!}$. First we construct a set $A \subseteq \Pi(N)$ containing $|G_n(k)|$ orders that perfectly covers each element of $G_n(k)$.

Since for each $i, j \in N$, $i \neq j$, and $S \subseteq N \backslash \{i, j\}$ with $|S| = k$ the set $P(S, \{i, j\})$ contains perfect coverings, it is trivial to obtain a set $A \subseteq \Pi(N)$ containing $|G_n(k)|$ orders that perfectly covers each element of $G_n(k)$. In particular, A can be obtained by choosing precisely one perfect covering from each element of $G_n(k)$.

Now assume that A perfectly covers each element in $\bigcup_{p=m}^{k} G_n(p)$ for some $m \in \{0, \ldots, k\}$. Obviously $m = k$ satisfies this property. Suppose that $P(S, \{i, j\}) \in G_n(m-1)$ is not perfectly covered by A. We will replace one order $\sigma \in A$ by an order $\bar{\sigma} \in \Pi(N) \backslash A$ to obtain the set $\bar{A} = (A \backslash \{\sigma\}) \cup \{\bar{\sigma}\}$. Our selection of σ and $\bar{\sigma}$ will be such that $\bar{A}$ perfectly covers one more element of $\bigcup_{p=m-1}^{k} G_n(p)$ than A does. In particular, $\bar{A}$ perfectly covers the same elements of $\bigcup_{p=m-1}^{k} G_n(p)$ as A, except for $P(S, \{i, j\}) \in G_n(m-1)$ which is only perfectly covered by $\bar{A}$, but not by A.

Without loss of generality assume that $i \in R_j$. This yields that if $\tau \in \Pi(N)$ perfectly covers $P(S, \{i, j\})$, then $\tau(|S|+1) = i$ and $\tau(|S|+2) = j$. Let B be the set of orders in A that begin with $S \cup \{i\}$ followed by j, i.e. $B = \Pi(S \cup \{i\}, \{j\}, N \backslash (S \cup \{i, j\})) \cap A$. We will replace an order $\sigma \in B$ with an order $\bar{\sigma} \in \Pi(N) \backslash A$.

Now first suppose that there is an order in B that is not a perfect covering of an element in $G_n(m-1)$, i.e. suppose there is a $\sigma \in B$ with $\sigma(|S|+1) \notin R_j$. Now interchange $\sigma(|S|+1)$ and i to obtain the order $\bar{\sigma}$. Note that $\bar{\sigma}$ and σ only differ in two positions, namely in position $\sigma^{-1}(i) \leq m$ and in position $|S|+1 = m$. This yields that $\bar{\sigma}$ perfectly covers the same elements of $\bigcup_{p=m}^{k} G_n(p)$ as σ. Furthermore, $\bar{\sigma}$ perfectly covers $P(S, \{i, j\})$. Because σ was not a perfect covering of an element of $G_n(m-1)$ it follows that $\bar{A} = (A \backslash \{\sigma\}) \cup \{\bar{\sigma}\}$ perfectly covers one more element of $\bigcup_{p=m-1}^{k} G_n(p)$ than A.

Now suppose that all orders in B are perfect coverings of elements in $G_n(m-1)$, i.e. suppose that $\tau(|S|+1) \in R_j$ for all $\tau \in B$. We will show that there are $\pi, \rho \in B$ with $\pi(|S|+1) = \rho(|S|+1) = h$ for some $h \in S$. That is, we show that $P((S \cup \{i\}) \backslash \{h\}, \{h, j\}) \in G_n(m-1)$ is perfectly covered twice by orders in B. If we then take $\sigma = \pi$ and obtain $\bar{\sigma}$ from σ by interchanging h and i, it follows that $\bar{A} = (A \backslash \{\sigma\}) \cup \{\bar{\sigma}\}$ still contains a perfect covering of $P((S \cup \{i\}) \backslash \{h\}, \{h, j\})$, namely ρ. Moreover, $\bar{A}$ perfectly covers $P(S, \{i, j\})$. Hence, $\bar{A}$ perfectly covers one more element of $\bigcup_{p=m-1}^{k} G_n(p)$ than A.

We will now show that there are orders $\pi, \rho \in B$ with $\pi(|S| + 1) = \rho(|S|+1)$. Note that, by supposition, $\tau(|S|+1) \in R_j$ for all $\tau \in B$. Because we have assumed that $P(S, \{i, j\})$ was not perfectly covered by an order in A, it follows that $\tau(|S| + 1) \neq i$ for all $\tau \in B$. Therefore, $\tau(|S| + 1) \in S$ for all $\tau \in B$. This implies that $\tau(|S| + 1) \in S \cap R_j$ for all $\tau \in B$. Hence, showing that there are orders $\pi, \rho \in B$ with $\pi(|S| + 1) = \rho(|S| + 1)$ boils down to showing that $|B| > |S \cap R_j|$.

First note that our assumption states that each element of $\bigcup_{p=m}^{k} G_n(p)$ is perfectly covered by A. This implies that $P(S \cup \{i\}, \{j, l\})$ is perfectly covered for all $l \in N \backslash (S \cup \{i, j\})$. Therefore $P(S \cup \{i\}, \{j, l\})$ is perfectly covered for all $l \in (N \backslash (S \cup \{i\})) \cap L_j$. Let $\tau \in A$ be a perfect covering of $P(S \cup \{i\}, \{j, l\})$ for some $l \in (N \backslash (S \cup \{i\})) \cap L_j$. Because of (P3) it follows that $j \in R_l$. Therefore $\tau(p) \in S \cup \{i\}$ for all $p \leq |S| + 1$, $\tau(|S| + 2) = j$ and $\tau(|S| + 3) = l$. We conclude that $\tau \in B$. This implies that

$$|B| \geq |(N \backslash (S \cup \{i\})) \cap L_j|. \tag{2.5}$$

It holds that $|(N \backslash (S \cup \{i\})) \cap L_j| + |(S \cup \{i\}) \cap L_j| = |L_j| = k$. Using (P2) it follows that $|(S \cup \{i\}) \cap L_j| + |(S \cup \{i\}) \cap R_j| = |S \cup \{i\}|$. From these two expressions we derive that

$$\begin{aligned} |(N \backslash (S \cup \{i\})) \cap L_j| &= k - |(S \cup \{i\}) \cap L_j| \\ &= k - |S \cup \{i\}| + |(S \cup \{i\}) \cap R_j| \\ &\geq |(S \cup \{i\}) \cap R_j|, \end{aligned} \tag{2.6}$$

where the inequality is satisfied because $k \geq m = |S| + 1$. Combining (2.5) and (2.6) yields

$$\begin{aligned} |B| &\geq |(S \cup \{i\}) \cap R_j| \\ &> |S \cap R_j|, \end{aligned}$$

where the strict inequality holds because $i \in S \cup \{i\}$ and $i \in R_j$.

So if we start with a set A containing $|G_n(k)|$ elements that perfectly covers each element of $\bigcup_{p=m}^{k} G_n(p)$, then we can find a set $\bar{A}$ that perfectly covers one more element of $\bigcup_{p=m-1}^{k} G_n(p)$ than A. This means that we can construct a set of orders that perfectly covers all elements of $\bigcup_{p=0}^{k} G_n(p)$. Now let $m \in \{k, \ldots, n-2\}$ be such that A perfectly covers all elements of $\bigcup_{p=0}^{m} G_n(p)$. Obviously, $m = k$ satisfies this property. Suppose that some $P(S, \{i,j\}) \in G_n(m+1)$ is not perfectly covered by A. It is now straightforward to show that there exists a set $\bar{A}$ that perfectly covers one more element of $\bigcup_{p=0}^{m+1} G_n(p)$ than A. It follows that there exists a set containing $|G_n(k)|$ orders that perfectly covers all elements of $\bigcup_{p=0}^{n-2} G_n(p)$. Because of Lemma 2.4.1 this set is complete. This concludes the proof. $\square$

The following example illustrates the possibility in the proof of Theorem 2.4.1 that there is a $\sigma \in B$ with $\sigma(|S|+1) \notin R_j$.

Example 2.4.3 Let $N = \{1, \ldots, 5\}$. Then $k = 2$. According to the proof of Theorem 2.4.1, we first need to find a set $A \subseteq \Pi(N)$ that perfectly covers each element of $G_5(2)$. This can be done by taking one perfect cover from each $P(S, \{i,j\}) \in G_5(2)$. For example, let $A =$

$$
\begin{array}{llllll}
\{ & (1,2,3,4,5), & (1,4,2,3,5), & (2,3,4,1,5), & (5,2,1,3,4), & (3,5,1,2,4), \\
 & (1,2,3,5,4), & (1,4,5,2,3), & (2,3,5,1,4), & (2,5,4,1,3), & (3,5,4,1,2), \\
 & (1,2,4,5,3), & (1,4,3,5,2), & (2,3,4,5,1), & (2,5,3,4,1), & (3,5,2,4,1), \\
 & (1,3,2,4,5), & (1,5,2,3,4), & (2,4,1,3,5), & (3,4,1,2,5), & (4,5,1,2,3), \\
 & (1,3,5,2,4), & (1,5,2,4,3), & (2,4,5,1,3), & (3,4,5,1,2), & (4,5,1,3,2), \\
 & (1,3,4,5,2), & (1,5,3,4,2), & (2,4,3,5,1), & (3,4,5,2,1), & (4,5,2,3,1)\ \}.
\end{array}
$$

It is straightforward to check that A indeed perfectly covers all elements of $G_5(2)$. However, not all elements of $G_5(1)$ are perfectly covered. For instance, A does not cover $P(\{5\}, \{3,4\})$, and hence it does not perfectly cover this set. We will now obtain a set $\bar{A}$ that perfectly covers $P(\{5\}, \{3,4\})$.

Let $S = \{5\}$, $i = 3$ and $j = 4$. Note that $i \in R_j$. Now define $B = \Pi(\{3,5\}, \{4\}, \{1,2\}) \cap A = \{(3,5,4,1,2)\}$. For $\sigma = (3,5,4,1,2) \in B$ it is satisfied that $\sigma(|S|+1) = 5 \notin R_4$. According to the proof we need to interchange $\sigma(|S|+1) = 5$ and $i = 3$. This yields $\bar{\sigma} = (5,3,4,1,2)$. Note that

$(5, 3, 4, 1, 2)$ perfectly covers $P(\{5\}, \{3, 4\})$. Now let $\bar{A} = (A \backslash \{(3, 5, 4, 1, 2)\}) \cup \{(5, 3, 4, 1, 2)\}$. Then $\bar{A}$ perfectly covers $P(\{5\}, \{3, 4\})$. $\diamond$

The following example illustrates the possibility that for all $\sigma \in B$, $\sigma(|S| + 1) \in R_j$.

Example 2.4.4 Let N, k, and A be the same as in Example 2.4.3. Although $(5, 2, 1, 3, 4) \in A$ covers $P(\{5\}, \{1, 2\}) \in G_5(1)$, it does not perfectly cover this set. Moreover, $P(\{5\}, \{1, 2\})$ is not perfectly covered by any order in A. Now let $S = \{5\}$, $i = 1$ and $j = 2$. Note that $i \in R_j$. Define $B = \Pi(\{1, 5\}, \{2\}, \{3, 4\}) \cap A = \{(1, 5, 2, 4, 3), (1, 5, 2, 3, 4)\}$. For all $\sigma \in B$ it is satisfied that $\sigma(|S| + 1) = 5 \in R_2$. So, $P(\{1\}, \{2, 5\})$ is perfectly covered twice by orders in B. Take $\sigma = (1, 5, 2, 4, 3) \in B$. Now interchange $\sigma(|S| + 1) = 5$ and $i = 1$ to obtain $\bar{\sigma} = (5, 1, 2, 4, 3)$ and let $\bar{A} = (A \backslash \{(1, 5, 2, 4, 3)\}) \cup \{(5, 1, 2, 4, 3)\}$. Then $\bar{A}$ still perfectly covers $P(\{1\}, \{2, 5\})$, and, moreover, $\bar{A}$ perfectly covers $P(\{5\}, \{1, 2\})$. $\diamond$

In the final part of this paper we deal with even $n \in \mathbb{N}$. Although the proof of the formula for even $n \in \mathbb{N}$ is similar to the proof for odd $n \in \mathbb{N}$, there are some differences between the two proofs. The main difference is that for even $n \in \mathbb{N}$ we have to redefine the concepts of right-hand side and left-hand side neighbours. The concept of perfect coverings remains the same, although it uses the adapted definitions of right-hand side and left-hand side neighbours.

Let $n \in \mathbb{N}$, $n \geq 4$, be even, $N = \{1, \ldots, n\}$ and $k = \frac{n-2}{2}$. For each $j \in N$, define the set of *right-hand side neighbours* R_j by

$$R_j = \{(j-1) \ mod \ n, \ldots, (j-k) \ mod \ n, (j-k-1) \ mod \ n\}$$

and the set of *left-hand side neighbours* L_j by

$$L_j = \{(j+1) \ mod \ n, \ldots, (j+k) \ mod \ n, (j+k+1) \ mod \ n\}.$$

The intuition of L_j and R_j is similar as for odd n. For convenience, we define $o_j = (j + k + 1) \ mod \ n$ for all $j \in N$. Intuitively, o_j is the player seated at the round table exactly opposite to player j. It is straightforward to show that $o_{(j+k+1)\ mod\ n} = j$ and that $o_j = (j-k-1)\ mod\ n$. The following properties can easily be verified.

(Q1) $L_j \cap R_j = \{o_j\}$;

(Q2) $L_j \cup R_j \cup \{j\} = N$;

(Q3) $i \in L_j$ if and only if $j \in R_i$;

(Q4) $i \in R_j$ or $j \in R_i$.

For each $j \in N$, player o_j is a member of L_j and R_j. This observation implies that (P1) does not hold anymore and that (P4) is only satisfied in a weakened version. Let $i, j \in N$ with $i \neq j$ and $S \subseteq N \backslash \{i, j\}$. Then $\sigma \in P(S, \{i, j\})$ is said to *perfectly cover* $P(S, \{i, j\})$ if $\sigma(|S|+1) \in R_{\sigma(|S|+2)}$. Note that if $i \neq o_j$, then half the number of orders in $P(S, \{i, j\})$ perfectly covers this set, while if $i = o_j$, then each order in $P(S, \{i, j\})$ perfectly covers this set. A set $A \subseteq \Pi(N)$ is called *perfect complete* if for each $i, j \in N$, $i \neq j$, and $S \subseteq N \backslash \{i, j\}$ there is a $\sigma \in A$ that perfectly covers $P(S, \{i, j\})$. Again, note that from Lemma 2.4.1 it follows that each perfect complete set is complete as well. The following theorem provides the formula for the minimum cardinality of a complete set for even n. Again we remark that the proof of this theorem is constructive in the sense that it provides a method to construct a perfect complete set of minimum cardinality.

Theorem 2.4.2 Let $n \in \mathbb{N}$ with $n \geq 4$ be even. Then

$$M_n = \frac{n!}{2(\frac{n-2}{2})!(\frac{n-2}{2})!}.$$

Proof: Let $k = \frac{n-2}{2}$. First we show that $M_n \geq \frac{n!}{2((\frac{n-2}{2})!)^2}$. First observe that $G_n(k)$ forms a partition of $\Pi(N)$. This implies that to cover all elements of $G_n(k)$ at least $|G_n(k)| = \binom{n}{k}\binom{n-k}{2} = \frac{n!}{k!k!2!} = \frac{n!}{2!((\frac{n-2}{2})!)^2}$ orders are needed. So, using Lemma 2.4.1, $M_n \geq \frac{n!}{2((\frac{n-2}{2})!)^2}$.

It remains to show that $M_n \leq \frac{n!}{2((\frac{n-2}{2})!)^2}$. The proof will be similar as for odd n. First construct a set $A \subseteq \Pi(N)$ containing $|G_n(k)|$ orders that perfectly covers each element of $G_n(k)$. Now assume that A perfectly covers each element of $\bigcup_{p=m}^{k} G_n(p)$ for some $m \leq k$, and suppose that $P(S, \{i, j\}) \in G_n(m-1)$ is not perfectly covered by A. Assume without loss of generality that $i \in R_j$ and let $B = \Pi(S \cup \{i\}, \{j\}, N \backslash (S \cup \{i, j\})) \cap A$.

If there is an order $\sigma \in B$ with $\sigma(|S|+1) \notin R_j$, i.e. if B contains an order that is not a perfect covering of some element in $G_n(m-1)$, then using the same technique as for odd n, it is straightforward to obtain a set $\bar{A}$ that perfectly covers one more element of $\bigcup_{p=m-1}^{k} G_n(p)$ than A.

So suppose that for all $\tau \in B$, $\tau(|S|+1) \in R_j$. That is, all orders in B are perfect coverings of some element of $G_n(m-1)$. Again, we will show that there are $\pi, \rho \in B$ with $\pi(|S|+1) = \rho(|S|+1) = h$ for some $h \in S$, i.e. that $P((S \cup \{i\}) \backslash \{h\}, \{h, j\}) \in G_n(m-1)$ is perfectly covered twice by orders in B. This boils down to showing that $|B| > |S \cap R_j|$. We distinguish between two cases to show this inequality.

`Case 1:` $o_j \in N \backslash (S \cup \{i\})$.

We assumed that each element of $\bigcup_{p=m}^{k} G_n(p)$ is perfectly covered by A. So $P(S \cup \{i\}, \{j, l\})$ is perfectly covered for all $l \in N \backslash (S \cup \{i, j\})$. Hence, $P(S \cup \{i\}, \{j, l\})$ is perfectly covered for all $l \in (N \backslash (S \cup \{i, o_j\})) \cap L_j$. Let $l \in (N \backslash (S \cup \{i, o_j\})) \cap L_j$ and let $\tau \in A$ be a perfect covering of $P(S \cup \{i\}, \{j, l\})$.

Because $l \neq o_j$, we conclude because of (Q1) that $l \notin R_j$. Because τ is a perfect covering it follows that $\tau(p) \in S \cup \{i\}$ for all $p \leq |S|+1$, $\tau(|S|+2) = j$ and $\tau(|S|+3) = l$. This implies that $\tau \in B$.

It follows that

$$\begin{aligned} |B| &\geq |(N \backslash (S \cup \{i, o_j\})) \cap L_j| \\ &= |(N \backslash (S \cup \{i\})) \cap L_j| - 1. \end{aligned}$$

The equality is satisfied since $o_j \in (N \backslash (S \cup \{i\})) \cap L_j$. Trivially, $|(N \backslash (S \cup \{i\})) \cap L_j| + |(S \cup \{i\}) \cap L_j| = k+1$. Because of $o_j \in N \backslash (S \cup \{i\})$, (Q1) and (Q2) it is satisfied that $|(S \cup \{i\}) \cap L_j| + |(S \cup \{i\}) \cap R_j| = |S \cup \{i\}|$. Hence,

$$\begin{aligned} |B| &\geq |(N \backslash (S \cup \{i\})) \cap L_j| - 1 \\ &= k + 1 - |(S \cup \{i\}) \cap L_j| - 1 \\ &= k + |(S \cup \{i\}) \cap R_j| - |S \cup \{i\}| \end{aligned}$$

$$\begin{aligned} &\geq |(S \cup \{i\}) \cap R_j| \\ &> |S \cap R_j|. \end{aligned}$$

The first inequality follows from $k \geq m = |(S \cup \{i\})|$. The strict inequality follows from $i \in S \cup \{i\}$ and $i \in R_j$.

`Case 2:` $o_j \in S \cup \{i\}$.

We assumed that each element of $\bigcup_{p=m}^{k} G_n(p)$ is perfectly covered by A. So $P(S \cup \{i\}, \{j, l\})$ is perfectly covered for all $l \in N \backslash (S \cup \{i, j\})$. Hence, $P(S \cup \{i\}, \{j, l\})$ is perfectly covered for all $l \in (N \backslash (S \cup \{i\})) \cap L_j$. Let $l \in (N \backslash (S \cup \{i\})) \cap L_j$ and let $\tau \in A$ be a perfect covering of $P(S \cup \{i\}, \{j, l\})$.

Because $o_j \in S \cup \{i\}$ it follows that $l \neq o_j$. This implies, using (Q1), that $l \notin R_j$. Hence, $\tau(p) \in S \cup \{i\}$ for all $p \leq |S|+1$, $\tau(|S|+2) = j$ and $\tau(|S|+3) = l$. It follows that $\tau \in B$. We conclude that $|B| \geq |(N \backslash (S \cup \{i\})) \cap L_j|$. It also holds that $|(N \backslash (S \cup \{i\})) \cap L_j| + |(S \cup \{i\}) \cap L_j| = k + 1$. Because of $o_j \in S \cup \{i\}$, (Q1) and (Q2), $|(S \cup \{i\}) \cap L_j| + |(S \cup \{i\}) \cap R_j| = |S \cup \{i\}| + 1$. Hence,

$$\begin{aligned} |B| &\geq |(N \backslash (S \cup \{i\})) \cap L_j| \\ &= k + 1 - |(S \cup \{i\}) \cap L_j| \\ &= k + 1 + |(S \cup \{i\}) \cap R_j| - (|S \cup \{i\}| + 1) \\ &\geq |(S \cup \{i\}) \cap R_j| \\ &> |S \cap R_j|. \end{aligned}$$

The first inequality follows from $k \geq m = |S \cup \{i\}|$. The strict inequality follows from $i \in S \cup \{i\}$ and $i \in R_j$.

Using the same argument as in the proof of Theorem 2.4.1 we can now obtain a perfect complete set, and hence a complete set, of size $G_n(k)$. $\square$

Theorems 2.4.1 and 2.4.2 provide formulas for the minimum number of marginal vectors needed to characterize convexity. For $n \in \{3, \ldots, 9\}$ these numbers are presented in Table 2.2. Note that M_n is relatively small for large n. Furthermore observe that the convergence of $\frac{M_n}{n!}$ is much faster than the convergence of F_n in Section 2.3.

n	3	4	5	6	7	8	9
$n!$	6	24	120	720	5040	40320	362880
$\frac{n!}{2}$	3	12	60	360	2520	20160	181440
M_n	3	12	30	90	210	560	1260
$\frac{M_n}{n!}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{24}$	$\frac{1}{72}$	$\frac{1}{288}$

Table 2.2: The minimum cardinality of complete sets.

2.5 Permutational convexity

In this section we introduce a refinement of permutational convexity. We show that the conditions of this refinement are still sufficient for the corresponding marginal vector to be a core element. Furthermore we show that permutational convexity can be redefined using a restricted set of inequalities. In particular we reduce the number of inequalities by a factor two. To conclude the section we will consider neighbours of permutationally convex orders, and we show that if an order is permutationally convex, then its last neighbour is permutationally convex as well.

Let $\sigma \in \Pi(N)$. Let $S \subseteq N$, $S \neq \emptyset$ and define $h(S) = \max\{j \in \{1, \ldots, |N|\} : \sigma(j) \notin S, S \not\subseteq [\sigma(j), \sigma]\}$ as the position of the highest ordered player outside S that precedes at least one player in S. We remark that $h(S)$ only exists if S is not a head[3] of σ.

We will call $v \in TU^N$ *weak permutationally convex with respect to* σ if for each $i, k \in \{0, \ldots, |N|-1\}$ with $i < k$ and for each $S \subseteq N \backslash [\sigma(k), \sigma]$ with $\sigma(k+1) \in S$ at least one of the following two inequalities is satisfied:

$$v(T) + v([\sigma(k), \sigma]) \leq v([\sigma(k), \sigma] \cup S) + v([\sigma(i), \sigma]) \tag{2.7}$$

$$v(T) + v([\sigma(h(S)), \sigma]) \leq v([\sigma(h(S)), \sigma] \cup T) + v(T \cap [\sigma(h(S)), \sigma]), \tag{2.8}$$

where $T = [\sigma(i), \sigma] \cup S$. We remark that (2.7) coincides with (2.1), but that the domain of (2.7) is more restricted because of the condition $\sigma(k+1) \in S$. Secondly we remark that if S is connected, then (2.7) and (2.8) coincide

[3] A head of σ is a coalition $S \subseteq N$ such that $S = \{\sigma(1), \ldots, \sigma(i)\}$ for some $i \in \{1, \ldots, |N|\}$.

since in this case $h(S) = k$. In the following example we illustrate weak permutational convexity.

Example 2.5.1 Let $N = \{1, 2, 3, 4, 5\}$, $v \in TU^N$ and $\sigma \in \Pi(N)$ be given by $\sigma(i) = i$ for each $i \in \{1, 2, 3, 4, 5\}$. Let $i = 1$, $k = 2$ and $S = \{3, 5\}$. Then $h(S) = 4$. Hence, the corresponding condition for weak permutational convexity is $v(\{1, 3, 5\}) + v(\{1, 2\}) \leq v(\{1, 2, 3, 5\}) + v(\{1\})$ or $v(\{1, 3, 5\}) + v(\{1, 2, 3, 4\}) \leq v(N) + v(\{1, 3\})$. Note that if $i = 1$, $k = 3$ and $S = \{4, 5\}$, then $h(S) = 3 = k$. So both (2.7) and (2.8) boil down to $v(\{1, 4, 5\}) + v(\{1, 2, 3\}) \leq v(N) + v(\{1\})$. $\diamond$

The following theorem shows that weak permutational convexity is sufficient for the corresponding marginal vector to be a core element.

Theorem 2.5.1 Let $\sigma \in \Pi(N)$. If $v \in TU^N$ is weak permutationally convex with respect to σ, then $m^\sigma(v) \in C(v)$.

Proof: Since marginal vectors are efficient by definition, we only need to show $\sum_{i \in W} m_i^\sigma(v) \geq v(W)$ for each $W \subseteq N$. We will first show that $\sum_{i \in W} m_i^\sigma(v) \geq v(W)$ for each $W \subseteq N$ consisting of only one component. Then we show the inequality for coalitions consisting of multiple components.

Let $W \subseteq N$ consist of only one component. If $\sigma(1) \in W$, then W is a head of σ, and trivially $\sum_{i \in W} m_i^\sigma(v) = v(W)$. So assume that $\sigma(1) \notin W$. Let $s \in N \backslash W$ be the highest ordered player in $N \backslash W$ preceding all players in W, and let $t \in W$ be the highest ordered player in W. Then

$$\sum_{i \in W} m_i^\sigma(v) = v([t, \sigma]) - v([s, \sigma]) \geq v(W).$$

The inequality is satisfied because both (2.7) and (2.8), with $i = 0$, $k = \sigma^{-1}(s)$ and $S = W$, coincide with $v(W) + v([s, \sigma]) \leq v([t, \sigma])$.

Now suppose that W consists of $a \geq 2$ components. Let $W_1, \ldots, W_a$ be these components. We assume that these components are ordered, i.e. if $i, k \in \{1, \ldots, a\}$ with $i < k$, then $W_i \subseteq [j, \sigma]$ for each $j \in W_k$. For each $i \in \{1, \ldots, a\}$, let t_i be the highest ordered player in W_i, and let s_i be the highest ordered player in $N\backslash W$ preceding all players in W_i. Define $s_1 = 0$ in case $\sigma(1) \in W_1$. We will now show by induction on $q-p$ that

$$\sum_{l=p}^{q} \sum_{i \in W_l} m_i^{\sigma}(v) \geq v([s_p, \sigma] \cup \bigcup_{l=p}^{q} W_l) - v([s_p, \sigma]) \tag{2.9}$$

for all $p, q \in \{1, \ldots, a\}$ with $p < q$. First we show the induction basis. So let $p, q \in \{1, \ldots, a\}$ with $p < q$ be such that $q - p = 1$. Then,

$$\begin{aligned}
\sum_{l=p}^{q} \sum_{i \in W_l} m_i^{\sigma}(v) &= v([t_p, \sigma]) - v([s_p, \sigma]) + v([t_q, \sigma]) - v([s_q, \sigma]) \\
&\geq v([t_p, \sigma] \cup W_q) - v([s_p, \sigma]) \\
&= v([s_p, \sigma] \cup W_p \cup W_q) - v([s_p, \sigma]).
\end{aligned}$$

The inequality is satisfied because both (2.7) and (2.8), with $i = \sigma^{-1}(t_p)$, $k = \sigma^{-1}(s_q)$ and $S = W_q$, coincide with $v([t_p, \sigma] \cup W_q) + v([s_q, \sigma]) \leq v([t_q, \sigma]) + v([t_p, \sigma])$.

Now assume, as the induction hypothesis, that $j \in \{1, \ldots, a-1\}$ is such that (2.9) is satisfied for all $p, q \in \{1, \ldots, a\}$ with $p < q$ and $q - p \leq j$. If $j = a - 1$, then we are done, so assume that $j < a - 1$. For the induction step, let $p^*, q^* \in \{1, \ldots, a\}$ be such that $p^* < q^*$ and $q^* - p^* = j + 1$. From (2.7) and (2.8) with $i = \sigma^{-1}(t_{p^*})$, $k = \sigma^{-1}(s_{p^*+1})$ and $S = \bigcup_{l=p^*+1}^{q^*} W_l$, we conclude that at least one of the following two inequalities is satisfied:

$$\begin{aligned}
&v([t_{p^*}, \sigma] \cup \bigcup_{l=p^*+1}^{q^*} W_l) + v([s_{p^*+1}, \sigma]) \\
\leq &v([s_{p^*+1}, \sigma] \cup \bigcup_{l=p^*+1}^{q^*} W_l) + v([t_{p^*}, \sigma]),
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
&v([t_{p^*}, \sigma] \cup \bigcup_{l=p^*+1}^{q^*} W_l) + v([s_{q^*}, \sigma]) \\
\leq &v([t_{q^*}, \sigma]) + v([t_{p^*}, \sigma] \cup \bigcup_{l=p^*+1}^{q^*-1} W_l).
\end{aligned} \tag{2.11}$$

First suppose that (2.10) is satisfied. Then,

$$\begin{aligned}
\sum_{l=p^*}^{q^*} \sum_{i \in W_l} m_i^\sigma(v) &= v([t_{p^*}, \sigma]) - v([s_{p^*}, \sigma]) + \sum_{l=p^*+1}^{q^*} \sum_{i \in W_l} m_i^\sigma(v) \\
&\geq v([t_{p^*}, \sigma]) - v([s_{p^*}, \sigma]) \\
&\quad + v([s_{p^*+1}, \sigma] \cup \bigcup_{l=p^*+1}^{q^*} W_l) - v([s_{p^*+1}, \sigma]) \\
&\geq v([t_{p^*}, \sigma] \cup \bigcup_{l=p^*+1}^{q^*} W_l) - v([s_{p^*}, \sigma]) \\
&= v([s_{p^*}, \sigma] \cup \bigcup_{l=p^*}^{q^*} W_l) - v([s_{p^*}, \sigma]).
\end{aligned}$$

The first inequality is satisfied since, according to the induction hypothesis, (2.9) is satisfied for each $p, q \in \{1, \ldots, a\}$ with $p < q$ and $q - p \leq j$. In particular, (2.9) is satisfied for the pair $p^* + 1, q^*$. The second inequality is due to (2.10).

Now suppose that (2.11) is satisfied. Then,

$$\begin{aligned}
\sum_{l=p^*}^{q^*} \sum_{i \in W_l} m_i^\sigma(v) &= \sum_{l=p^*}^{q^*-1} \sum_{i \in W_l} m_i^\sigma(v) + v([t_{q^*}, \sigma]) - v([s_{q^*}, \sigma]) \\
&\geq v([s_{p^*}, \sigma] \cup \bigcup_{l=p^*}^{q^*-1} W_l) - v([s_{p^*}, \sigma]) \\
&\quad + v([t_{q^*}, \sigma]) - v([s_{q^*}, \sigma]) \\
&= v([t_{p^*}, \sigma] \cup \bigcup_{l=p^*+1}^{q^*-1} W_l) - v([s_{p^*}, \sigma]) \\
&\quad + v([t_{q^*}, \sigma]) - v([s_{q^*}, \sigma]) \\
&\geq v([t_{p^*}, \sigma] \cup \bigcup_{l=p^*+1}^{q^*} W_l) - v([s_{p^*}, \sigma]) \\
&= v([s_{p^*}, \sigma] \cup \bigcup_{l=p^*}^{q^*} W_l) - v([s_{p^*}, \sigma]).
\end{aligned}$$

The first inequality is satisfied since, according to the induction hypothesis, (2.9) is satisfied for each $p, q \in \{1, \ldots, a\}$ with $p < q$ and $q - p \leq j$. In particular, (2.9) is satisfied for the pair $p^*, q^* - 1$. The second inequality is due to (2.11).

We conclude that (2.9) is satisfied for each $p, q \in \{1, \ldots, a\}$ with $p < q$. It follows from (2.9), with $p = 1$ and $q = a$, that

$$\sum_{i \in W} m_i^\sigma(v) \geq v([s_1, \sigma] \cup \bigcup_{l=1}^{a} W_l) - v([s_1, \sigma]) = v([s_1, \sigma] \cup W) - v([s_1, \sigma]). \quad (2.12)$$

If $s_1 = 0$, then we are done. So assume that $s_1 \neq 0$. In order to show that $\sum_{i \in W} m_i^\sigma(v) \geq v(W)$, we need to prove one more assertion with induction. To be more precise, we show for each $r \in \{1, \ldots, a\}$ that

$$\sum_{l=1}^{r} \sum_{i \in W_l} m_i^\sigma(v) \geq v(\bigcup_{l=1}^{r} W_l). \quad (2.13)$$

This implies, using $r = a$, that

$$\sum_{i \in W} m_i^\sigma(v) = \sum_{l=1}^{a} \sum_{i \in W_l} m_i^\sigma(v) \geq v(\bigcup_{l=1}^{a} W_l) = v(W).$$

It remains to show that (2.13) is indeed satisfied for all $r \in \{1, \ldots, a\}$. For the induction basis, let $r = 1$. Then (2.13) is satisfied since we already showed this inequality for coalitions consisting of one component only.

Now assume, as the induction hypothesis, that $j \in \{1, \ldots, a\}$ is such that (2.13) is satisfied for all $r \in \{1, \ldots, j\}$. If $j = a$, then we are done, so assume that $j < a$. For the induction step, let $r^* = j + 1$. From (2.7) and (2.8), with $i = 0$, $k = \sigma^{-1}(s_1)$ and $S = \bigcup_{l=1}^{r^*} W_l$, we conclude that at least one of the following two inequalities is satisfied:

$$\begin{aligned} v(\bigcup_{l=1}^{r^*} W_l) + v([s_1, \sigma]) &\leq v([s_1, \sigma] \cup \bigcup_{l=1}^{r^*} W_l) && (2.14) \\ v(\bigcup_{l=1}^{r^*} W_l) + v([s_{r^*}, \sigma]) &\leq v([t_{r^*}, \sigma]) + v(\bigcup_{l=1}^{r^*-1} W_l). && (2.15) \end{aligned}$$

First suppose that (2.14) is satisfied. Then,

$$\sum_{l=1}^{r^*} \sum_{i \in W_l} m_i^\sigma(v) \geq v([s_1, \sigma] \cup \bigcup_{l=1}^{r^*} W_l) - v([s_1, \sigma]) \geq v(\bigcup_{l=1}^{r^*} W_l).$$

The first inequality follows from (2.9) with $p = 1$ and $q = r^*$ and the second from (2.14).

Now suppose that (2.15) is satisfied. Then

$$\begin{aligned}\sum_{l=1}^{r^*}\sum_{i\in W_l} m_i^\sigma(v) &= \sum_{l=1}^{r^*-1}\sum_{i\in W_l} m_i^\sigma(v) + v([t_{r^*},\sigma]) - v([s_{r^*},\sigma]) \\ &\geq v(\bigcup_{l=1}^{r^*-1} W_l) + v([t_{r^*},\sigma]) - v([s_{r^*},\sigma]) \\ &\geq v(\bigcup_{l=1}^{r^*} W_l).\end{aligned}$$

The first inequality is due to our induction hypothesis that (2.13) is satisfied for all $r \in \{1, \ldots, j\}$. In particular, (2.13) is satisfied for $r = r^* - 1$. The second inequality holds because of (2.15). □

We remark that weak permutational convexity is a weaker condition than permutational convexity. For weak permutational convexity one needs to check a pair of inequalities for each $i, k \in \{0, \ldots, |N|-1\}$ with $i < k$ and each $S \subseteq N\backslash[\sigma(k), \sigma]$ with $\sigma(k+1) \in S$. In fact, for each $i, k \in \{0, \ldots, |N|-1\}$ with $i < k$, there are precisely $2^{|N|-k-1}$ choices of $S \subseteq N\backslash[\sigma(k), \sigma]$ such that $\sigma(k+1) \in S$. Therefore, weak permutational convexity requires the checking of

$$\begin{aligned}\sum_{i=0}^{|N|-2}\sum_{k=i+1}^{|N|-1} 2^{|N|-k-1} &= \sum_{i=0}^{|N|-2}[2^{|N|-i-1} - 1] \\ &= 2^{|N|} - 2 - (|N| - 1) \\ &= 2^{|N|} - |N| - 1\end{aligned}$$

pairs of inequalities. The following example is meant to illustrate that weak permutational convexity is not a necessary condition for the corresponding marginal vector to be a core element.

Example 2.5.2 Consider $v \in TU^N$ with $N = \{1, 2, 3, 4\}$, $v(S) = 0$ if $S \in 2^N\backslash\{\{1, 2\}, \{2, 4\}, \{1, 2, 3\}, N\}$ and $v(\{1, 2\}) = v(\{2, 4\}) = v(\{1, 2, 3\}) = v(N) = 1$. Let $\sigma \in \Pi(N)$ be given by $\sigma(i) = i$ for each $i \in \{1, 2, 3, 4\}$.

Observe that (N, v) is not weak permutationally convex with respect to σ, since the corresponding condition is not satisfied for $i = 0$, $k = 1$ and $S = \{2, 4\}$. However, $m^\sigma(v) = (0, 1, 0, 0) \in C(v)$. $\diamond$

In the upcoming proposition we consider a set of inequalities and show that these inequalities are equivalent to permutational convexity.

Proposition 2.5.1 Let $v \in TU^N$. Then (N, v) is permutationally convex with respect to $\sigma \in \Pi(N)$ if and only if

$$v([\sigma(i), \sigma] \cup S) + v([\sigma(k), \sigma]) \leq v([\sigma(k), \sigma] \cup S) + v([\sigma(i), \sigma]) \qquad (2.16)$$

for all $i, k \in \{0, \ldots, |N| - 1\}$ with $i + 1 = k$ and $S \subseteq N \backslash [\sigma(k), \sigma]$ with $S \neq \emptyset$.

Proof: Observe that (2.16) coincides with (2.1), but that the domain of (2.16) is restricted by the extra condition $i + 1 = k$. Hence, the "only if" part follows directly from the definition of permutational convexity. Therefore we only show the "if" part. Assume that (2.16) is satisfied for all $i, k \in \{0, \ldots, |N| - 1\}$ with $i + 1 = k$ and $S \subseteq N \backslash [\sigma(k), \sigma]$ with $S \neq \emptyset$. We need to show that (2.16) is satisfied for all $i, k \in \{0, \ldots, |N| - 1\}$ with $i < k$ and $S \subseteq N \backslash [\sigma(k), \sigma]$ with $S \neq \emptyset$. We use backwards induction on i.

For the induction basis let $i = |N| - 2$, $k = |N| - 1$ and $S \subseteq N \backslash [\sigma(|N| - 1), \sigma]$ with $S \neq \emptyset$. In this case, (2.16) is satisfied by assumption since $i + 1 = k$.

For the induction hypothesis we assume that for some $i^* \in \{1, \ldots, |N| - 2\}$ we have shown that (2.16) is satisfied for all $i, k \in \{0, \ldots, |N| - 1\}$ with $i^* \leq i$, $i < k$, and $S \subseteq N \backslash [\sigma(k), \sigma]$ with $S \neq \emptyset$.

For the induction step, let $i, k \in \{0, \ldots, |N| - 1\}$ be such that $i = i^* - 1$, $i < k$ and $S \subseteq N \backslash [\sigma(k), \sigma]$ with $S \neq \emptyset$. If $i + 1 = k$, then our inequality is satisfied by assumption so assume that $i + 1 < k$. From our induction hypothesis it follows that (2.16) is satisfied for $\bar{i} = i^*$, $\bar{k} = k$ and S. Hence,

$$v([\sigma(i^*), \sigma] \cup S) + v([\sigma(k), \sigma]) \leq v([\sigma(k), \sigma] \cup S) + v([\sigma(i^*), \sigma]). \quad (2.17)$$

We also know from our initial assumption that (2.16) is satisfied for $\bar{i} = i$, $\bar{k} = i^*$ and S, since $\bar{i} + 1 = \bar{k}$. This yields

$$v([\sigma(i), \sigma] \cup S) + v([\sigma(i^*), \sigma]) \leq v([\sigma(i^*), \sigma] \cup S) + v([\sigma(i), \sigma]). \quad (2.18)$$

Now adding (2.17) and (2.18) gives

$$v([\sigma(i),\sigma]\cup S)+v([\sigma(k),\sigma])\leq v([\sigma(k),\sigma]\cup S)+v([\sigma(i),\sigma]),$$

which is precisely the inequality we needed to show. □

Let $v\in TU^N$ and $\sigma\in\Pi(N)$. According to Proposition 2.5.1, permutational convexity of σ requires the checking of precisely $2^{|N|-k}-1$ inequalities for each pair $i,k\in\{0,\ldots,|N|-1\}$ with $i+1=k$. So permutational convexity can be checked by considering

$$\begin{aligned}\sum_{k=1}^{|N|-1}[2^{|N|-k}-1] &= 2^{|N|}-2-(|N|-1)\\ &= 2^{|N|}-|N|-1\end{aligned}$$

inequalities. In particular, Proposition 2.5.1 reduces the number of permutational convexity inequalities by a factor two, approximately.

We conclude this section by considering neighbours of permutationally convex orders. In particular, we present a condition such that a neighbour of a permutationally convex order is permutationally convex as well.

Proposition 2.5.2 Let $v\in TU^N$. Let $\sigma\in\Pi(N)$ be a permutationally convex order for (N,v) and let $l\in\{1,\ldots,|N|-1\}$. If

$$v([\sigma_l(i),\sigma_l]\cup S)+v([\sigma_l(k),\sigma_l])\leq v([\sigma_l(k),\sigma_l]\cup S)+v([\sigma_l(i),\sigma_l]),\quad(2.19)$$

for $i=l-1$, $k=l$ and all $S\subseteq N\backslash[\sigma_l(l+1),\sigma_l]$ with $S\neq\emptyset$, and for $i=l$, $k=l+1$ and all $S\subseteq N\backslash[\sigma_l(l+1),\sigma_l]$ with $S\neq\emptyset$, then σ_l is permutationally convex for (N,v).

Proof: According to Proposition 2.5.1 showing that σ_l is permutationally convex boils down to showing (2.19) for all $i,k\in\{0,\ldots,|N|-1\}$ with $i+1=k$ and $S\subseteq N\backslash[\sigma_l(k),\sigma_l]$ with $S\neq\emptyset$. We distinguish between three cases.

`Case 1:` $i\leq l-2$ or $i\geq l+1$.

In this case $[\sigma_l(i), \sigma_l] = [\sigma(i), \sigma]$ and $[\sigma_l(k), \sigma_l] = [\sigma(k), \sigma]$. Let $S \subseteq N \backslash [\sigma_l(k), \sigma_l] = N \backslash [\sigma(k), \sigma]$ with $S \neq \emptyset$. Since σ is permutationally convex for (N, v) we conclude that (2.1) is satisfied for i, $k = i+1$ and S. Observe that (2.1) coincides with (2.19).

`Case 2:` $i = l - 1$.

In this case $k = i + 1 = l$. We need to show that (2.19) holds for all $S \subseteq N \backslash [\sigma_l(k), \sigma_l]$ with $S \neq \emptyset$. First note that if $\sigma_l(k+1) \notin S$, then $S \subseteq N \backslash [\sigma_l(k+1), \sigma_l] = N \backslash [\sigma_l(l+1), \sigma_l]$. In this case (2.19) is satisfied by assumption. So now suppose that $\sigma_l(k+1) \in S$. Let $T = S \backslash \{\sigma_l(k+1)\}$. Since σ is permutationally convex for (N, v) it follows that (2.1) holds with $\bar{i} = i$, $\bar{k} = k$ and $\bar{S} = \{\sigma(k+1)\}$, i.e.

$$v([\sigma(i), \sigma] \cup \{\sigma(k+1)\}) + v([\sigma(k), \sigma]) \leq v([\sigma(k), \sigma] \cup \{\sigma(k+1)\}) + v([\sigma(i), \sigma]),$$

which is equivalent to

$$v([\sigma(i), \sigma] \cup \{\sigma(k+1)\}) + v([\sigma(k), \sigma]) \leq v([\sigma(k+1), \sigma]) + v([\sigma(i), \sigma]). \quad (2.20)$$

Now if $T \neq \emptyset$, then (2.1) is satisfied for $\hat{i} = i + 1 = k$, $\hat{k} = k + 1$ and $\hat{S} = T$, since σ is permutationally convex by assumption. This implies

$$v([\sigma(k), \sigma] \cup T) + v([\sigma(k+1), \sigma]) \leq v([\sigma(k+1), \sigma] \cup T) + v([\sigma(k), \sigma]). \quad (2.21)$$

Note that (2.21) is trivially satisfied in case $T = \emptyset$ as well. Now adding (2.20) and (2.21) yields

$$\begin{aligned} & v([\sigma(k), \sigma] \cup T) + v([\sigma(i), \sigma] \cup \{\sigma(k+1)\}) \\ \leq\ & v([\sigma(k+1), \sigma] \cup T) + v([\sigma(i), \sigma]). \end{aligned} \quad (2.22)$$

Observe that $[\sigma(k), \sigma] \cup T = [\sigma_l(i), \sigma_l] \cup S$, $[\sigma(i), \sigma] \cup \{\sigma(k+1)\} = [\sigma_l(k), \sigma_l]$, $[\sigma(k+1), \sigma] \cup T = [\sigma_l(k), \sigma_l] \cup S$ and $[\sigma(i), \sigma] = [\sigma_l(i), \sigma_l]$. This shows that (2.22) coincides with (2.19).

`Case 3:` $i = l$.

In this case $k = i + 1 = l + 1$. Let $S \subseteq N \backslash [\sigma_l(l+1), \sigma_l]$ with $S \neq \emptyset$. Now (2.19) is satisfied by assumption. □

Proposition 2.5.2 easily constitutes the following corollary. This corollary shows that if one order is permutationally convex, then its last neighbour is permutationally convex as well. Hence, for each game an even number of orders is permutationally convex.

Corollary 2.5.1 Let $v \in TU^N$. If $\sigma \in \Pi(N)$ is permutationally convex for (N, v), then $\sigma_{|N|-1}$ is permutationally convex for (N, v) as well.

Proof: According to Proposition 2.5.2 it is sufficient to show that

$$\begin{aligned} & v([\sigma_{|N|-1}(i), \sigma_{|N|-1}] \cup S) + v([\sigma_{|N|-1}(k), \sigma_{|N|-1}]) \\ \leq \;\; & v([\sigma_{|N|-1}(k), \sigma_{|N|-1}] \cup S) + v([\sigma_{|N|-1}(i), \sigma_{|N|-1}]) \end{aligned}$$

is satisfied for $i = |N| - 2$, $k = |N| - 1$ and all $S \subseteq N \backslash [\sigma_{|N|-1}(|N|), \sigma_{|N|-1}]$ with $S \neq \emptyset$, and for $i = |N| - 1$, $k = |N|$ and all $S \subseteq N \backslash [\sigma_{|N|-1}(|N|), \sigma_{|N|-1}]$ with $S \neq \emptyset$. However, if $S \subseteq N \backslash [\sigma_{|N|-1}(|N|), \sigma_{|N|-1}]$, then $S = \emptyset$, so the condition is an empty one. $\square$

Chapter 3

Tree-component additive games

3.1 Introduction

In cooperative game theory, the cooperative possibilities between players are often severely restricted. For instance, in assignment games (cf. Shapley and Shubik (1972)), bridge games (cf. Shubik (1971)) and neighbour games (cf. Klijn, Vermeulen, Hamers, Solymosi, Tijs, and Villar (2003)) only small coalitions play an essential role. In sequencing games (see Chapter 6) it is often assumed that only the so-called connected coalitions have full cooperative possibilities, and in Myerson (1977) the cooperative possibilities between players are restricted by an exogenously given graph.

We will also assume that the cooperative possibilities between players are restricted by an exogenously given graph. In particular, we consider the case where this graph is a tree, and we assume that associated games are superadditive. The resulting class of cooperative games is the class of tree-component additive games. It can be shown that tree-component additive games satisfy several nice game theoretical properties. For instance, it is proved in LeBreton, Owen, and Weber (1991) that tree-component additive games have non-empty cores. Furthermore, it is shown in Potters and Reijnierse (1995) that the core coincides with the bargaining set and that the kernel only consists of the nucleolus. In Solymosi, Aarts, and

Driessen (1998) a primal and in Kuipers, Solymosi, and Aarts (2000) a dual type algorithm is presented for the efficient computation of the nucleolus.

This chapter, which is based on Van Velzen, Hamers, and Solymosi (2004), mainly focuses on core stability. Several related properties like largeness of the core and exactness are also considered. The main tool for this study will be covering families. A covering family is, bluntly speaking, a minimal set of connected coalitions that covers the entire player set. By associating an inequality with each covering family we derive a characterisation of largeness of the core in tree-component additive games. We will also characterise exactness by applying a result of Schmeidler (1972) to tree-component additive games.

To obtain a sufficient condition for core stability we restrict the set of covering families. In particular, we introduce the set of basic covering families, consisting of those covering families for which at most one coalition does not contain a leaf. The corresponding restricted set of inequalities is proved to be sufficient for core stability. In fact, we will show that this set of inequalities is sufficient for a refinement of extendibility, called essential extendibility. Essential extendibility is the property that each core element of each subgame corresponding to an essential coalition can be extended to a core element. By showing that essential extendibility is a sufficient condition for core stability of any TU game, we are able to prove that the basic covering family inequalities give rise to a sufficient condition for core stability.

Finally, we apply the results we derived for tree-component additive games to the special situation where the tree is a chain. Furthermore we show that for the corresponding class of chain-component additive games largeness of the core is equivalent to exactness. Moreover, we prove that essential extendibility is equivalent to core stability, and both concepts are characterised in terms of linear equalities and inequalities. The necessity of these linear equalities and inequalities is shown using a dual approach. First we appoint a certain subset of the imputation set and we show, using a variant of Farkas' Lemma, that this subset contains an undominated imputation outside the core if and only if all vectors from a related polyhedron satisfy a well-chosen linear inequality. We then decompose each member of this polyhedron into a sum of three types of basis vectors. Finally we use these basis vectors to show that the well-chosen linear inequality is indeed satisfied for each member of the polyhedron.

The remainder of this chapter is organised as follows. Section 3.2 formally introduces the class of tree-component additive games. In Section 3.3 largeness of the core as well as exactness are characterised for tree-component additive games. Section 3.4 introduces essential extendibility and shows its relation to core stability. Furthermore a sufficient condition for essential extendibility is provided. The special case of chain-component additive games is treated in Section 3.5. Finally, in Section 3.6 some lemmas are proved that are needed for the characterisation of essential extendibility and core stability.

3.2 Tree-component additive games

In this section we formally introduce tree-component additive games, and we develop some notation. We begin the section with the definition of core stability, exactness, largeness and extendibility.

Let $v \in TU^N$ and $x, y \in I(v)$. Then x is said to *dominate* y via coalition $S \subseteq N$ if $\sum_{i \in S} x_i \leq v(S)$ and $x_i > y_i$ for all $i \in S$. The core is called *stable* if for each imputation y outside the core there is a core element x and a coalition $S \subseteq N$ such that x dominates y via S. A game $v \in TU^N$ is said to be *exact* if for each $S \subseteq N$ there is an $x \in C(v)$ with $\sum_{i \in S} x_i = v(S)$. The core is *large* if for each $x \in U(v)$ there is a $y \in C(v)$ with $y_i \leq x_i$ for each $i \in N$. Finally, a game is *extendible* if each core element of each subgame can be extended to a core element of (N, v). In other words, a game $v \in TU^N$ is extendible if for each $T \subseteq N$ and each $x \in C(v_T)$ there exists a $y \in C(v)$ with $y_i = x_i$ for each $i \in T$.

In Sharkey (1982) it is shown that largeness of the core is a sufficient condition for core stability. It is proved in Kikuta and Shapley (1986) that extendibility is necessary for largeness of the core and sufficient for core stability. If a game has a non-empty core, and if all subgames have non-empty cores as well, then largeness is sufficient for exactness (Sharkey (1982)), as well as extendibility is sufficient for exactness (Biswas, Parthasarathy, Potters, and Voorneveld (1999)). In general, exactness and core stability do not imply one another (cf. Biswas, Parthasarathy, Potters, and Voorneveld (1999) and Van Gellekom, Potters, and Reijnierse (1999)).

Let $G = (N, E)$ be a tree. The subgraph of G with respect to coalition $S \subseteq N$ is given by (S, E_S), where $\{i, j\} \in E_S$ if $\{i, j\} \in E$ and $i, j \in S$. For each $S \subseteq N$, the subgraph (S, E_S) consists of several maximally connected subtrees. Denote the set of vertex sets of these maximally connected subtrees by $\mathcal{C}(S)$. A game $v \in TU^N$ is called *tree-component additive with respect to* G if it is superadditive, and if $v(S) = \sum_{T \in \mathcal{C}(S)} v(T)$ for each $S \subseteq N$. That is, the worth of each coalition is equal to the sum of the worths of its maximally connected components. If $G = (N, E)$ is a chain, then (N, v) is called *chain-component additive*. In the remainder of this chapter we assume, without loss of generality, that tree-component additive games are zero-normalised, i.e. if $v \in TU^N$ is a tree-component additive game, then $v(\{i\}) = 0$ for each $i \in N$.

The following example illustrates the concept of tree-component additive games.

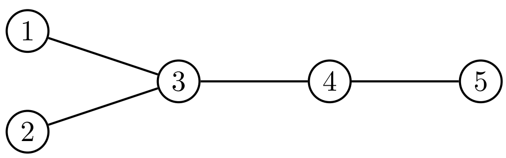

Figure 3.1: A tree $G = (N, E)$.

Example 3.2.1 Let $G = (N, E)$ be the tree depicted in Figure 3.1. Then $\{1, 2, 4, 5\}$ consists of three maximally connected components. In particular, $\mathcal{C}(\{1, 2, 4, 5\}) = \{\{1\}, \{2\}, \{4, 5\}\}$. If $v \in TU^N$ is tree-component additive with respect to G, then $v(\{1, 2, 4, 5\}) = v(\{1\}) + v(\{2\}) + v(\{4, 5\})$. $\diamond$

In tree-component additive games only the connected coalitions play a vital role in determining the shape of the core and the upper-core. To be more precise, let $G = (N, E)$ be a tree and $v \in TU^N$ a tree-component additive game with respect to G. Denote the set of connected coalitions with respect to G by $\mathcal{S}$. Hence, $S \in \mathcal{S}$ if and only if $|\mathcal{C}(S)| = 1$. Then $C(v) = \{x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S)$ for each $S \in \mathcal{S}\}$, and $U(v) = \{x \in \mathbb{R}^N : \sum_{i \in S} x_i \geq v(S)$ for each $S \in \mathcal{S}\}$.

The following two results are shown in LeBreton, Owen, and Weber (1991). The first result states that each balanced collection being a subset of $\mathcal{S}$ necessarily contains a partition. The second result shows that tree-component additive games have non-empty cores. Note this result can be straightforwardly proved by combining Lemma 3.2.1 with Theorem 1.2.1 while using superadditivity of tree-component additive games.

Lemma 3.2.1 (LeBreton, Owen, and Weber (1991)) Let $G = (N, E)$ be a tree. Let $B \subseteq \mathcal{S}$ be a balanced collection. Then B contains a partition of N as a subset.

Theorem 3.2.1 (LeBreton, Owen, and Weber (1991)) Let $G = (N, E)$ be a tree and let $v \in TU^N$ be tree-component additive with respect to G. Then $C(v) \neq \emptyset$.

Because tree-component additive games have non-empty cores, it easily follows that forest-component additive games have non-empty cores as well. This implies that subgames of tree-component additive games have non-empty cores, since each subgame of a tree-component additive game is forest-component additive.

3.3 Largeness and exactness

In this section we introduce covering families and we associate with each covering family an inequality. We show that these inequalities provide a characterisation of largeness of the core. Furthermore we use a result of Schmeidler (1972) to characterise exactness.

First we introduce covering families. Let $G = (N, E)$ be a tree. Then $\mathcal{T} \subseteq \mathcal{S}$ is called a *covering family* if $\mathcal{T} \neq \{N\}$, and if

(A1) $\bigcup_{T \in \mathcal{T}} T = N$;

(A2) $\bigcup_{S \in \mathcal{T} \setminus \{T\}} S \neq N$ for all $T \in \mathcal{T}$;

(A3) if $S, T \in \mathcal{T}$ with $S \cap T = \emptyset$, then $S \cup T \notin \mathcal{S}$.

Requirement (A1) states that every player is covered by $\mathcal{T}$, and (A2) states that every element of $\mathcal{T}$ is necessary to cover the complete player set. Finally, requirement (A3) implies that two members of a covering family are either intersecting, or non-adjacent. The following example illustrates covering families.

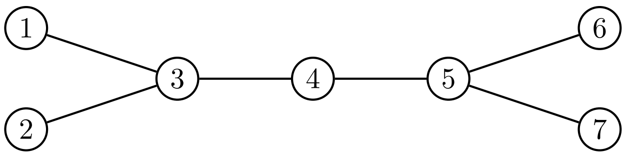

Figure 3.2: A tree $G = (N, E)$.

Example 3.3.1 Let $G = (N, E)$ be the tree depicted in Figure 3.2. Note that $\mathcal{T} = \{\{1, 3, 4\}, \{2, 3, 4\}, \{4, 5, 6\}, \{4, 5, 7\}\}$ forms a covering family. Indeed, N is covered by $\mathcal{T}$ and each element of $\mathcal{T}$ is needed to cover N. Furthermore, $S \cap T \neq \emptyset$ for each $S, T \in \mathcal{T}$, which shows that (A3) is satisfied as well. The sets $\{\{1, 2, 3\}, \{1, 2, 3, 4, 5\}, \{5, 6, 7\}\}$ and $\{\{1, 3, 4\}, \{2\}, \{4, 5, 6, 7\}\}$ do not form covering families, since conditions (A2) and (A3) are violated, respectively. $\diamond$

With each covering family we will associate a so-called covering family inequality. These covering family inequalities will provide a characterisation of largeness of the core in tree-component additive games. Before we introduce covering family inequalities, we first need to introduce $\mathcal{T}$-balancing vectors.

Let $\mathcal{T} \subseteq \mathcal{S}$ be a covering family. Define $\lambda_i(\mathcal{T}) = |\{T \in \mathcal{T} : i \in T\}| - 1$ for all $i \in N$ and $W(\mathcal{T}) = \{i \in N : \lambda_i(\mathcal{T}) > 0\}$. That is, $\lambda_i(\mathcal{T})$ is the number of times that i is covered more than once, and $W(\mathcal{T})$ consists of the players covered more than once. The vector $y : \mathcal{S} \to \mathbb{R}_+$ is called $\mathcal{T}$*-balancing* if

$\sum_{S\in\mathcal{S}} y_S e(S) = \lambda(\mathcal{T})$. Note that if y is $\mathcal{T}$-balancing, then $\sum_{T\in\mathcal{T}} e(T) = e(N) + \sum_{S\in\mathcal{S}} y_S e(S)$. The set of $\mathcal{T}$-balancing vectors is denoted by $B(\mathcal{T})$. We remark that $B(\mathcal{T})$ is non-empty and compact. The *covering family inequality* associated to $\mathcal{T}$ is

$$\sum_{T\in\mathcal{T}} v(T) \leq v(N) + \max\{\sum_{S\in\mathcal{S}} y_S v(S) : y \in B(\mathcal{T})\}. \tag{3.1}$$

We illustrate the concept of covering family inequalities in the following example.

Example 3.3.2 Consider Figure 3.2. In Example 3.3.1 it is shown that $\mathcal{T} = \{\{1,3,4\},\{2,3,4\},\{4,5,6\},\{4,5,7\}\}$ forms a covering family. Observe that $\lambda_i(\mathcal{T}) = 0$ for $i \in \{1,2,6,7\}$, $\lambda_3(\mathcal{T}) = \lambda_5(\mathcal{T}) = 1$ and $\lambda_4(\mathcal{T}) = 3$. Furthermore, $W(\mathcal{T}) = \{3,4,5\}$. Finally, observe that, for instance, $y : \mathcal{S} \to \mathbb{R}_+$ with $y_{\{4\}} = y_{\{3,4\}} = y_{\{4,5\}} = 1$ and $y_S = 0$ if $S \in \mathcal{S}\backslash\{\{4\},\{3,4\},\{4,5\}\}$ is $\mathcal{T}$-balancing. The covering family inequality associated to $\mathcal{T}$ is:

$$\begin{aligned} & v(\{1,3,4\}) + v(\{2,3,4\}) + v(\{4,5,6\}) + v(\{4,5,7\}) \\ \leq\ & v(N) + \max\{\sum_{S\in\mathcal{S}} y_S v(S) : y \in B(\mathcal{T})\}. \end{aligned} \qquad \diamond$$

Before we present our characterisation of largeness of the core, we first find an alternative expression for $\max\{\sum_{S\in\mathcal{S}} y_S v(S) : y \in B(\mathcal{T})\}$. Let $G = (N,E)$ be a tree and $\mathcal{T}$ a covering family. For notational convenience, define $\mathcal{S}(W(\mathcal{T})) = \{S \in \mathcal{S} : S \subseteq W(\mathcal{T})\}$. Let $y \in B(\mathcal{T})$. Then $y \geq 0$ and $\sum_{S\in\mathcal{S}} y_S e(S) = \lambda(\mathcal{T})$. This implies that $y_S = 0$ for all $S \notin \mathcal{S}(W(\mathcal{T}))$. Therefore,

$$\begin{aligned} B(\mathcal{T}) = \{y \in \mathbb{R}^{\mathcal{S}} : \ & y \geq 0, y_S = 0 \text{ if } S \notin \mathcal{S}(W(\mathcal{T})), \\ & \sum_{S\in\mathcal{S}(W(\mathcal{T})), i\in S} y_S = \lambda_i(\mathcal{T}) \text{ for all } i \in W(\mathcal{T})\}. \end{aligned}$$

We conclude that

$$\max\{\sum_{S\in\mathcal{S}} y_S v(S) : y \in B(\mathcal{T})\} \tag{3.2}$$

$$= \max\{\sum_{S\in\mathcal{S}(W(\mathcal{T}))} y_S v(S) : \sum_{\substack{S\in\mathcal{S}(W(\mathcal{T})):\\ i\in S}} y_S = \lambda_i(\mathcal{T}) \text{ for all } i \in W(\mathcal{T}), y \geq 0\}.$$

$$= \min\{\sum_{i\in W(\mathcal{T})} \lambda_i(\mathcal{T}) x_i : \sum_{i\in S} x_i \geq v(S) \text{ for all } S \in \mathcal{S}(W(\mathcal{T}))\}. \tag{3.3}$$

The second equality follows from Theorem 1.2.3.

The following theorem provides a characterisation of largeness of the core in terms of covering family inequalities. For the proof of this theorem it is convenient to show two lemmas first. The first lemma uses the following definition. Let $x \in \mathbb{R}^N$. Define $\mathcal{S}(x) = \{S \in \mathcal{S} : \sum_{i\in S} x_i = v(S)\}$, i.e. $\mathcal{S}(x)$ are the connected coalitions that are tight at x. Lemma 3.3.1 shows that if at an allocation in the upper-core two disjoint coalitions with a connected union are tight, then this union is tight as well. Lemma 3.3.2 shows that if a subset of $\mathcal{S}$ is closed under union of disjoint elements, and if it covers all players, then it contains a covering family as as subset.

Lemma 3.3.1 Let $G = (N, E)$ be a tree. Let $v \in TU^N$ be tree-component additive with respect to G and let $x \in U(v)$. If $A, B \in \mathcal{S}(x)$ with $A \cap B = \emptyset$ and $A \cup B \in \mathcal{S}$, then $A \cup B \in \mathcal{S}(x)$.

Proof: Since $x \in U(v)$, $\sum_{j\in A\cup B} x_j \geq v(A \cup B)$. Observe

$$\sum_{j\in A\cup B} x_j = \sum_{j\in A} x_j + \sum_{j\in B} x_j = v(A) + v(B) \leq v(A \cup B),$$

where the inequality holds because of superadditivity. Since $\sum_{j\in A\cup B} x_j = v(A \cup B)$ and $A \cup B \in \mathcal{S}$ we conclude that $A \cup B \in \mathcal{S}(x)$. □

Lemma 3.3.2 Let $G = (N, E)$ be a tree. Let $\mathcal{V} \subseteq \mathcal{S}$. If for all $A, B \in \mathcal{V}$ with $A \cap B = \emptyset$ and $A \cup B \in \mathcal{S}$ it holds that $A \cup B \in \mathcal{V}$, and if $\bigcup_{T\in\mathcal{V}} T = N$, then $\mathcal{V}$ contains a covering family as a subset.

Proof: Let $X = \{T \in \mathcal{V} : T \not\subseteq S \text{ for all } S \in \mathcal{V}\backslash\{T\}\}$ be the set of maximal elements in $\mathcal{V}$. We will show that X contains a subset X^p that satisfies (A1) and (A2). Subsequently we show that X^p also satisfies (A3). Hence, X^p is a covering family.

We recursively construct a subset $X^p \subseteq X$ that satisfies (A1) and (A2). First note that X satisfies (A1), since the assumption $\bigcup_{T \in \mathcal{V}} T = N$ implies $\bigcup_{T \in X} T = N$. If X satisfies (A2) as well, then we are done, so assume that (A2) is violated. Clearly there is a $T^1 \in X$ with $\bigcup_{T \in X \backslash \{T^1\}} T = N$. This means that $X^1 = X \backslash \{T^1\}$ still satisfies (A1). If X^1 now violates (A2), then we can apply the same reasoning to obtain an $X^2 \subsetneq X^1$ that satisfies (A1). We recursively obtain an $X^p \subseteq X$, $p \geq 1$, that satisfies (A1) and (A2). Note that this recursion finishes in a finite number of steps since the cardinality of X is finite. It remains to show that (A3) is satisfied by X^p.

If $A, B \in X^p$ are such that $A \cap B = \emptyset$ and $A \cup B \in \mathcal{S}$, then $A \cup B \in \mathcal{V}$ by assumption, contradicting the fact that A and B are maximal elements in $\mathcal{V}$. So (A3) is satisfied as well. $\square$

We are now ready to characterise largeness in terms of covering family inequalities.

Theorem 3.3.1 Let $G = (N, E)$ be a tree and let $v \in TU^N$ be tree-component additive with respect to G. Then $C(v)$ is large if and only if all covering family inequalities are satisfied.

Proof: First we show the "if" part. Assume that all covering family inequalities are satisfied. Let $x \in U(v)$. If $\sum_{i \in N} x_i = v(N)$, then $x \in C(v)$ and we are done, so assume that $\sum_{i \in N} x_i > v(N)$. We need to show the existence of a $y \in C(v)$ with $y \leq x$. Instead, we show the existence of an $x^1 \in U(v)$ with $x^1_j < x_j$ for some $j \in N$ and $x^1_i = x_i$ for all $i \in N \backslash \{j\}$. Observe that $\sum_{i \in N} x^1_i < \sum_{i \in N} x_i$. We then argue that, by proceeding in the same way, we can construct an $x^p \in C(v)$ with $x^p \leq x$ in $p \leq |N|$ steps.

First we show, by contradiction, the existence of a $j \in N$ with $j \notin \bigcup_{T \in \mathcal{S}(x)} T$. Suppose that $j \in \bigcup_{T \in \mathcal{S}(x)} T$ for each $j \in N$. From Lemma 3.3.1 it follows that if $A, B \in \mathcal{S}(x)$ are such that $A \cap B = \emptyset$ and $A \cup B \in \mathcal{S}$, then $A \cup B \in \mathcal{S}(x)$. Hence, we can apply Lemma 3.3.2 with $\mathcal{V} = \mathcal{S}(x)$ to conclude that $\mathcal{S}(x)$ contains a covering family. Let $\mathcal{T} \subseteq \mathcal{S}(x)$ be a covering family. Since $T \in \mathcal{S}(x)$ for each $T \in \mathcal{T}$, we have $\sum_{i \in T} x_i = v(T)$ for all $T \in \mathcal{T}$. This yields

$$\begin{aligned}
\sum_{T\in\mathcal{T}}\sum_{i\in T} x_i &= \sum_{T\in\mathcal{T}} v(T)\\
&\le v(N)+\max\{\sum_{S\in\mathcal{S}} y_S v(S) : y\in B(\mathcal{T})\}\\
&\le v(N)+\max\{\sum_{S\in\mathcal{S}} y_S \sum_{i\in S} x_i : y\in B(\mathcal{T})\}\\
&< \sum_{i\in N} x_i+\max\{\sum_{S\in\mathcal{S}} y_S \sum_{i\in S} x_i : y\in B(\mathcal{T})\}\\
&= \sum_{i\in N} x_i+\sum_{i\in N}\lambda_i(\mathcal{T})x_i.
\end{aligned}$$

The first inequality holds since we assumed that all covering family inequalities are satisfied, and the second because $x\in U(v)$ gives $\sum_{i\in S} x_i \ge v(S)$ for each $S\in\mathcal{S}$. The strict inequality is due to our assumption that $\sum_{i\in N} x_i > v(N)$. The last equality is due to the definition of $\mathcal{T}$-balancing vectors and the definition of $\lambda(\mathcal{T})$.

We obtained that $\sum_{T\in\mathcal{T}}\sum_{i\in T} x_i < \sum_{i\in N} x_i+\sum_{i\in N}\lambda_i(\mathcal{T})x_i$. However, since $\mathcal{T}$ is a covering family, $\sum_{T\in\mathcal{T}}\sum_{i\in T} x_i = \sum_{i\in N} x_i+\sum_{i\in N}\lambda_i(\mathcal{T})x_i$. Because of this contradiction we conclude that it cannot occur that $\bigcup_{T\in\mathcal{S}(x)} T = N$.

Now let $j\in N$ be such that $j\notin\bigcup_{T\in\mathcal{S}(x)} T$. Define $\varepsilon=\min\{\sum_{i\in S} x_i - v(S) : j\in S, S\in\mathcal{S}\}$. Observe that $\varepsilon>0$, since $\sum_{i\in S} x_i > v(S)$ for all $S\in\mathcal{S}$ with $j\in S$. Define x^1 by $x^1_i = x_i$ for each $i\in N\backslash\{j\}$ and $x^1_j = x_j-\varepsilon$. Note that $x^1\in U(v)$. Indeed, for all $T\in\mathcal{S}$ with $j\notin T$ we have $\sum_{i\in T} x^1_i = \sum_{i\in T} x_i \ge v(T)$. Furthermore, $\sum_{i\in T} x^1_i = \sum_{i\in T} x_i - \varepsilon \ge v(T)$ for all $T\in\mathcal{S}$ with $j\in T$, since $\varepsilon=\min\{\sum_{i\in S} x_i - v(S) : j\in S, S\in\mathcal{S}\}\le \sum_{i\in T} x_i - v(T)$. We conclude that $x^1\in U(v)$, $x^1\le x$, and $\sum_{i\in N} x^1_i < \sum_{i\in N} x_i$. Also note that $\mathcal{S}(x)\subsetneq\mathcal{S}(x^1)$, since all coalitions that are tight at x are also tight at x^1, while at x^1 at least one more coalition is tight. In fact, $\bigcup_{T\in\mathcal{S}(x)} T \subsetneq \bigcup_{T\in\mathcal{S}(x^1)} T$, because at x^1 a coalition containing player j is tight.

If $\sum_{i\in N} x^1_i = v(N)$, then we are done. If $\sum_{i\in N} x^1_i > v(N)$, then we can apply the same procedure to obtain an $x^2\in U(v)$ with $x^2\le x^1\le x$, and $\sum_{i\in N} x^2_i < \sum_{i\in N} x^1_i < \sum_{i\in N} x_i$. Recursively we obtain a sequence

$x^1, \ldots, x^p$ with $x^m \in U(v)$ for all $m \in \{1, \ldots, p\}$, $x^p \leq x^{p-1} \leq \ldots \leq x$, and $\sum_{i\in N} x_i^p < \sum_{i\in N} x_i^{p-1} < \ldots < \sum_{i\in N} x_i$. Observe that by definition of ε it is satisfied that $\mathcal{S}(x) \subsetneq \mathcal{S}(x^1) \subsetneq \ldots \subsetneq \mathcal{S}(x^p)$. Since the set of players not covered by the families $\mathcal{S}(x), \mathcal{S}(x^1), \ldots, \mathcal{S}(x^p)$ is strictly shrinking, it follows that we construct an $x^p \in C(v)$ in $p \leq |N|$ steps. Hence, $C(v)$ is large.

We now prove the "only if"-part. Let $\mathcal{T}$ be a covering family for which the corresponding inequality is violated. We will construct an $x \in U(v)$ with $z \notin C(v)$ for each $z \leq x$ with $\sum_{i\in N} z_i = v(N)$. This shows that $C(v)$ is not large and therefore the covering family inequality associated with $\mathcal{T}$ is necessary.

Let $x^* \in \mathbb{R}^{W(\mathcal{T})}$ be a solution of (3.3). Define $x \in \mathbb{R}^N$ by $x_i = x_i^*$ if $i \in W(\mathcal{T})$ and $x_i = v(N)$ if $i \in N \backslash W(\mathcal{T})$. We claim that $x \in U(v)$. Indeed, if $S \in \mathcal{S}$ is such that $S \subseteq W(\mathcal{T})$, then $\sum_{i\in S} x_i \geq v(S)$ is satisfied by definition of (3.3). If $S \in \mathcal{S}$ is such that $S \not\subseteq W(\mathcal{T})$, then $\sum_{i\in S} x_i \geq v(N) + \sum_{i\in S\cap W(\mathcal{T})} x_i^* \geq v(N) \geq v(S)$. The first inequality is satisfied because (N, v) is zero-normalised and superadditive, which implies that $v(N) \geq 0$. The second inequality follows from $x_i^* \geq v(\{i\}) = 0$, for each $i \in W(\mathcal{T})$. The last inequality follows from superadditivity and the fact that (N, v) is zero-normalised.

Let $z \leq x$ be such that $\sum_{i\in N} z_i = v(N)$. We show that $z \notin C(v)$. Observe that

$$\begin{aligned}
\sum_{T\in\mathcal{T}} \sum_{i\in T} z_i &= \sum_{i\in N} z_i + \sum_{i\in W(\mathcal{T})} \lambda_i(\mathcal{T}) z_i \\
&= v(N) + \sum_{i\in W(\mathcal{T})} \lambda_i(\mathcal{T}) z_i \\
&\leq v(N) + \sum_{i\in W(\mathcal{T})} \lambda_i(\mathcal{T}) x_i \\
&= v(N) + \sum_{i\in W(\mathcal{T})} \lambda_i(\mathcal{T}) x_i^* \\
&= v(N) + \max\{\sum_{S\in\mathcal{S}} y_S v(S) : y \in B(\mathcal{T})\} \\
&< \sum_{T\in\mathcal{T}} v(T).
\end{aligned}$$

The first equality is satisfied by definition of $\lambda(\mathcal{T})$ and $W(\mathcal{T})$ and the second because $\sum_{i\in N} z_i = v(N)$. The first inequality is due to $z \leq x$. The fourth equality holds because the optimal value of (3.2) coincides with the optimal value of (3.3), and because we assumed that x^* is an optimal solution of (3.3). Finally, the strict inequality is satisfied because we assumed that the covering family inequality associated with $\mathcal{T}$ is violated.

From $\sum_{T\in\mathcal{T}}\sum_{i\in T} z_i < \sum_{T\in\mathcal{T}} v(T)$, we conclude there is a $T \in \mathcal{T}$ with $\sum_{i\in T} z_i < v(T)$. Therefore, $z \notin C(v)$. □

The next example illustrates the "only if" part of the proof of Theorem 3.3.1.

Example 3.3.3 Let $G = (N, E)$ be the tree depicted in Figure 3.1 on page 52. Let $v \in TU^N$ be the tree-component additive game with respect to G given by $v(\{i\}) = 0$ for all $i \in N$, and

$$v(S) = \begin{cases} 0, & \text{if } S = \{1,3\}, \{4,5\}; \\ 1, & \text{if } S = \{3,4\}\ ; \\ 2, & \text{if } S = \{1,2,3\}, \{1,2,3,4\}, \{1,3,4\}, \{1,3,4,5\}, \{2,3\} \\ & \qquad \{2,3,4\}, \{2,3,4,5\}, \{3,4,5\}; \\ 4, & \text{if } S = N. \end{cases}$$

The worths of the disconnected coalitions can be obtained from the tree-component additivity of (N, v).

Obviously, $\mathcal{T} = \{\{1,3,4\}, \{2,3\}, \{3,4,5\}\}$ is a covering family with $\lambda(\mathcal{T}) = (0,0,2,1,0)$ and $W(\mathcal{T}) = \{3,4\}$. Observe that

$$\max\{\sum_{S\in\mathcal{S}} y_S v(S) : y \in B(\mathcal{T})\} = v(\{3\}) + v(\{3,4\}) = 1.$$

Since, $v(\{1,3,4\}) + v(\{2,3\}) + v(\{3,4,5\}) = 6 > 5 = v(N) + v(\{3\}) + v(\{3,4\})$, we conclude that the covering family inequality associated with $\mathcal{T}$ is violated.

For showing that $C(v)$ is not large, we construct an $x \in U(v)$ with $z \notin C(v)$ for every $z \leq x$ with $\sum_{i\in N} z_i = v(N)$. First consider

$$\begin{aligned} & \min\{\sum_{i\in W(\mathcal{T})} \lambda_i(\mathcal{T}) x_i : \sum_{i\in S} x_i \geq v(S) \text{ for all } S \in \mathcal{S}(W(\mathcal{T}))\} \\ = \ & \min\{2x_3 + x_4 : x_3 + x_4 \geq 1, x_3 \geq 0, x_4 \geq 0\}. \end{aligned}$$

The optimal solution for this linear programming problem is given by $(x_3^*, x_4^*) = (0, 1)$. Now let $x = (v(N), v(N), x_3^*, x_4^*, v(N)) = (4, 4, 0, 1, 4) \in U(v)$. Let $z \leq x$ be such that $\sum_{i \in N} z_i = v(N) = 4$. Then

$$\begin{aligned}
\sum_{T \in \mathcal{T}} \sum_{i \in T} z_i &= \sum_{i \in N} z_i + 2z_3 + z_4 \\
&\leq v(N) + 2x_3 + x_4 \\
&= 5 \\
&< v(\{1,3,4\}) + v(\{2,3\}) + v(\{3,4,5\}).
\end{aligned}$$

We conclude that $z_1 + z_3 + z_4 < 2 = v(\{1,3,4\})$, $z_2 + z_3 < 2 = v(\{2,3\})$, or $z_3 + z_4 + z_5 < 2 = v(\{3,4,5\})$. This shows $z \notin C(v)$. $\diamond$

The remainder of this section is dedicated to exactness. First we recall a result proved in Schmeidler (1972). Then we apply this result on tree-component additive games. We conclude with an example that shows that largeness is not a necessary condition for exactness in tree-component additive games.

Theorem 3.3.2 (Schmeidler (1972)) A game $v \in TU^N$ is exact if and only if for each $T \subseteq N$, and each $y : 2^N \to \mathbb{R}_+$ with $\sum_{S \subseteq N, S \neq N} y_S e(S) = y_N e(N) + e(T)$,

$$\sum_{S \subseteq N, S \neq N} y_S v(S) \leq y_N v(N) + v(T).$$

In tree-component additive games all essential coalitions are connected. This implies that if we apply Theorem 3.3.2 to tree-component additive games, we only need to consider connected coalitions on the left-hand side of the equality and inequality.

Theorem 3.3.3 Let $G = (N, E)$ be a tree and let $v \in TU^N$ be tree-component additive with respect to G. Then (N, v) is exact if and only if for each $T \subseteq N$, and each $y : \mathcal{S} \to \mathbb{R}_+$ with $\sum_{S \in \mathcal{S} \setminus \{N\}} y_S e(S) = y_N e(N) + e(T)$,

$$\sum_{S \in \mathcal{S} \setminus \{N\}} y_S v(S) \leq y_N v(N) + v(T).$$

We conclude this section with an example of a tree-component additive game that is exact, but without a large core. This shows that largeness is not equivalent to exactness for tree-component additive games. However, we will show in Section 3.5 that these concepts are equivalent on the class of chain-component additive games.

Example 3.3.4 Let $v \in TU^N$ be the tree-component additive game without a large core from Example 3.3.3. Observe that $x^1 = (0,2,0,2,0)$, $x^2 = (0,0,4,0,0)$, $x^3 = (1,1,1,0,1)$, $x^4 = (2,0,2,0,0)$ and $x^5 = (0,0,2,0,2)$ are core elements. It is straightforward to check that for each $S \subseteq N$ there is an $i \in \{1,\dots,5\}$ with $\sum_{j\in S} x^i_j = v(S)$. Thus, (N,v) is exact. $\diamond$

3.4 Core stability

In this section we focus on core stability in tree-component additive games. We will introduce a refinement of extendibility, called essential extendibility, and show that this refinement is a sufficient condition for core stability of any TU game. Finally, we derive a sufficient condition for essential extendibility of tree-component additive games.

We begin this section with the introduction of connected marginal vectors. Subsequently we show that connected marginal vectors provide core elements in tree-component additive games. Let $G = (N,E)$ be a tree. An order $\sigma \in \Pi(N)$ is said to be *connected with respect to* G if $N \backslash [\sigma(j),\sigma]$ is a single component in G for each $j \in \{1,\dots,|N|\}$. A marginal vector is called *connected* if the corresponding order is connected. The next example illustrates connected orders.

Example 3.4.1 Consider Figure 3.1 on page 52. Then $\sigma = (1,2,3,4,5)$ is a connected order. However, the order $\tau = (1,3,2,4,5)$ is not connected, since for $j = 2$, $N \backslash [\tau(2),\tau] = \{2,4,5\}$ consists of two components with respect to G, namely $\{2\}$ and $\{4,5\}$. $\diamond$

Connected marginal vectors allow for a nice expression for the payoff of connected coalitions. Before we show this expression, we first develop some notation. Let $G = (N,E)$ be a tree, $\sigma \in \Pi(N)$ a connected order and $S \in \mathcal{S}$ a connected coalition. Define $M(S) = \{j \in N \backslash S : \{i,j\} \in E \text{ for some } i \in S\}$ to be the set of players of $N \backslash S$ that are adjacent to a player in S. Let $M^\sigma(S) = \{j \in M(S) : S \not\subseteq [j,\sigma]\}$ consist of those players of $M(S)$ that precede a player in S with respect to σ. Note that the connectedness of σ implies that $|M(S) \backslash M^\sigma(S)| \leq 1$ for each $S \in \mathcal{S}$. Finally, let $C_j \in \mathcal{C}(N \backslash S)$ be the maximally connected component of $N \backslash S$ containing player $j \in M(S)$. We illustrate the notation we just introduced.

Example 3.4.2 Consider Figure 3.2, which is depicted on page 54. Clearly, $\sigma = (1,2,3,4,5,6,7)$ is connected. If $S = \{4,5\}$, then $M(S) = \{3,6,7\}$ and $M^\sigma(S) = \{3\}$. Furthermore $C_3 = \{1,2,3\}$. $\diamond$

The following lemma describes the payoff of connected coalitions at connected marginal vectors.

Lemma 3.4.1 Let $G = (N,E)$ be a tree and $v \in TU^N$ be tree-component additive with respect to G. Let $\sigma \in \Pi(N)$ be connected with respect to G. If $S \in \mathcal{S}$, then

$$\sum_{i \in S} m_i^\sigma(v) = v(S \cup \bigcup_{j \in M^\sigma(S)} C_j) - \sum_{j \in M^\sigma(S)} v(C_j).$$

Proof: Let $k \in S$ be such that $S \subseteq [k,\sigma]$. Now first observe that

$$\begin{aligned} \sum_{i \in S} m_i^\sigma(v) \quad &= \quad \sum_{i \in [k,\sigma]} m_i^\sigma(v) &(3.4)\\ &\quad - \sum_{j \in M^\sigma(S)} \sum_{i \in [k,\sigma] \cap C_j} m_i^\sigma(v) &(3.5)\\ &\quad - \sum_{j \in M(S) \backslash M^\sigma(S)} \sum_{i \in [k,\sigma] \cap C_j} m_i^\sigma(v). &(3.6) \end{aligned}$$

Let $j \in M^\sigma(S)$. Note that $j \in [k,\sigma]$ by definition of $M^\sigma(S)$. Let $h \in S$ be such that $\{h,j\} \in E$. We claim that $j \in [h,\sigma]$. Indeed, if $h = k$, then clearly $j \in [k,\sigma] = [h,\sigma]$. If $h \neq k$ and $j \notin [h,\sigma]$, then $N \backslash [h,\sigma]$ contains at least two components, namely a component containing player j and a component containing k. This contradicts the connectedness of σ.

Now let $i \in C_j$. We claim $i \in [j,\sigma]$. Indeed, if $i \notin [j,\sigma]$, then $N \backslash [j,\sigma]$ contains at least two components, namely a component containing player i and a component containing player k. Again, this contradicts the connectedness of σ. We conclude that $i \in [h,\sigma]$ for each $i \in C_j$. This implies that $C_j \subseteq [k,\sigma]$ for each $j \in M^\sigma(S)$. Furthermore, using the tree-component additivity of (N,v), we conclude that $\sum_{i \in C_j} m_i^\sigma(v) = v(C_j)$ for each $j \in M^\sigma(S)$. Hence, expression (3.5) can be rewritten as

$$-\sum_{j \in M^\sigma(S)} \sum_{i \in [k,\sigma] \cap C_j} m_i^\sigma(v) = -\sum_{j \in M^\sigma(S)} v(C_j). \tag{3.7}$$

Furthermore, for expression (3.4) it is satisfied that

$$\begin{aligned}\sum_{i \in [k,\sigma]} m_i^\sigma(v) &= v(S \cup \bigcup_{j \in M^\sigma(S)} C_j \cup \bigcup_{j \in M(S) \setminus M^\sigma(S)} (C_j \cap [k,\sigma])) \\ &= v(S \cup \bigcup_{j \in M^\sigma(S)} C_j) + v(\bigcup_{j \in M(S) \setminus M^\sigma(S)} (C_j \cap [k,\sigma])). \end{aligned} \tag{3.8}$$

The first equality holds by definition of marginal vectors. The second equality follows from tree-component additivity of (N,v) and the fact that $S \cup \bigcup_{j \in M^\sigma(S)} C_j$ and $\bigcup_{j \in M(S) \setminus M^\sigma(S)} (C_j \cap [k,\sigma])$ are not connected.

Now finally note that the tree-component additivity of (N,v) implies that

$$-\sum_{j \in M(S) \setminus M^\sigma(S)} \sum_{i \in [k,\sigma] \cap C_j} m_i^\sigma(v) = -v(\bigcup_{j \in M(S) \setminus M^\sigma(S)} (C_j \cap [k,\sigma])). \tag{3.9}$$

The lemma now follows by substituting (3.7), (3.8) and (3.9) into (3.4), (3.5) and (3.6). □

Example 3.4.3 Consider Example 3.4.2. Let $v \in TU^N$ be tree-component additive with respect to G. Let $S = \{4,5\}$. Then, $\sum_{i \in S} m_i^\sigma(v) =$ $v(S \cup \bigcup_{j \in M^\sigma(S)} C_j) - \sum_{j \in M^\sigma(S)} v(C_j) = v(\{1,2,3,4,5\}) - v(\{1,2,3\})$. ◇

The following theorem shows that for tree-component additive games connected marginal vectors are core elements.

Theorem 3.4.1 Let $G = (N,E)$ be a tree and let $v \in TU^N$ be tree-component additive with respect to G. If $\sigma \in \Pi(N)$ is connected with respect to G, then $m^\sigma(v) \in C(v)$.

Proof: Let $\sigma \in \Pi(N)$ be a connected with respect to G. Since $m^\sigma(v)$ is efficient by definition, we only need to show that $\sum_{i\in S} m_i^\sigma(v) \geq v(S)$ for each $S \subseteq N$. In fact, since (N, v) is tree-component additive, it is sufficient to show coalition rationality for connected coalitions. Let $S \in \mathcal{S}$. Straightforwardly,

$$\sum_{i\in S} m_i^\sigma(v) = v(S \cup \bigcup_{j\in M^\sigma(S)} C_j) - \sum_{j\in M^\sigma(S)} v(C_j) \geq v(S),$$

where the equality is satisfied due to Lemma 3.4.1 and the inequality because tree-component additive games are superadditive. □

The remainder of this section is dedicated to core stability and essential extendibility. First we introduce essential extendibility and we prove that for any TU game essential extendibility is a sufficient condition for core stability.

A game $v \in TU^N$ is called *essential extendible* if for all essential coalitions $S \subseteq N$ and all $x \in C(v_S)$ there is a $y \in C(v)$ with $y_i = x_i$ for all $i \in S$. Obviously, largeness of the core and extendibility are sufficient conditions for essential extendibility. The following theorem shows that essential extendibility is sufficient for core stability.

Theorem 3.4.2 Let $v \in TU^N$ be essential extendible. Then $C(v)$ is stable.

Proof: Let $x \in I(v)\backslash C(v)$. Let $S \subseteq N$ be such that $\sum_{i\in S} x_i < v(S)$ and $\sum_{i\in T} x_i \geq v(T)$ for each $T \subsetneq S$. We claim that S is essential. Indeed, suppose S is inessential. Then there is a partition P of S with $\sum_{U\in P} v(U) \geq v(S) > \sum_{i\in S} x_i = \sum_{U\in P}\sum_{i\in U} x_i$. This implies that at least one coalition in P is dissatisfied with the payoff at x, which contradicts our choice of S. Now let $y \in \mathbb{R}^S$ be given by $y_i = x_i + \frac{v(S)-\sum_{j\in S} x_j}{|S|}$ for each $i \in S$. This yields, $y \in C(v_S)$. Because (N, v) is essential extendible, and because S is essential, there is a $z \in C(v)$ with $z_i = y_i$ for each $i \in S$. Observe that z dominates x via S. □

We will now develop a sufficient condition for essential extendibility in tree-component additive games. In order to do so, we introduce basic covering families. Let $G = (N, E)$ be a tree and let $\mathcal{T}$ be a covering family. Then $\mathcal{T}$ is called *basic* if at most one coalition in $\mathcal{T}$ does not contain a leaf of G. If $\mathcal{T}$ is basic and contains a coalition without a leaf, then this coalition is called the *central coalition*.

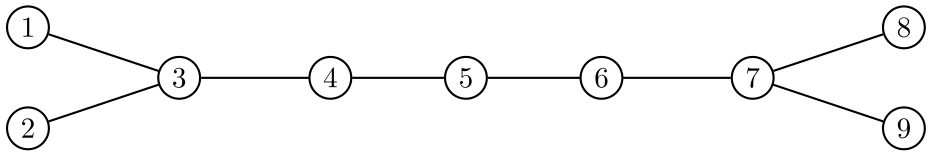

Figure 3.3: A tree $G = (N, E)$.

Example 3.4.4 Observe that the covering family from Example 3.3.1 is basic, since each covering family element contains a leaf. Now consider the tree depicted in Figure 3.3. The covering family $\{\{1, 2, 3\}, \{3, 4, 5, 6\}, \{6, 7, 8, 9\}\}$ is a basic covering family with central coalition $\{3, 4, 5, 6\}$. The covering family $\{\{1, 2, 3\}, \{3, 4, 5\}, \{5, 6, 7\}, \{7, 8, 9\}\}$ is not a basic covering family since both $\{3, 4, 5\}$ and $\{5, 6, 7\}$ do not contain leaves. $\diamond$

The following theorem describes a sufficient condition for essential extendibility in terms of basic covering family inequalities.

Theorem 3.4.3 Let $G = (N, E)$ be a tree and $v \in TU^N$ be tree-component additive with respect to G. If for each basic covering family without a central coalition and for each basic covering family with an essential central coalition the associated inequality is satisfied, then (N, v) is essential extendible.

Proof: Let $T \subseteq N$ be essential and let $x \in C(v_T)$. We will extend x to $y \in C(v)$. First we introduce some notation and definitions. Since (N, v) is tree-component additive it follows that $T \in \mathcal{S}$. For notational convenience, we will write $M = \{j_1, \ldots, j_{|M|}\}$ instead of $M(T)$. For each $j \in M$, let $\sigma^j \in \Pi(C_j)$ be such that $N \backslash [\sigma^j(l), \sigma^j]$ is connected for all $l \in \{1, \ldots, |C_j|\}$. Hence, σ^j is a connected order for the subtree (C_j, E_{C_j}) and $\sigma^j(|C_j|) = j$. With each σ^j we associate a partial marginal vector in the following straightforward way. Let $m_i^{\sigma^j}(v) = v([i, \sigma^j]) - v([i, \sigma^j] \backslash \{i\})$ for each $j \in M$ and $i \in C_j$.

Observe, using Theorem 3.4.1 and our definition of σ^j, that $\sum_{i\in S} m_i^{\sigma^j}(v) \geq v(S)$ for each $j \in M$ and $S \subseteq C_j$. Therefore, $\sum_{i\in S} m_i^{\sigma^j}(v) \geq v(S)$ for each $j \in M$ and $S \subseteq C_j \backslash \{j\}$.

Now define $y_i = x_i$ if $i \in T$, $y_i = m_i^{\sigma^j}(v)$ if $i \in C_j\backslash\{j\}$ for some $j \in M$, and recursively define

$$y_i = \max\{v(U) - \sum_{k\in U\backslash\{i\}} y_k : U \subseteq T \cup \bigcup_{l=1}^{p} C_{j_l}, i \in U, U \in \mathcal{S}\}$$

if $i = j_p$ for some $p \in \{1, \ldots, |M|\}$. First we show that $y \in U(v)$. Subsequently we show that $\sum_{i\in N} y_i = v(N)$, which proves $y \in C(v)$. Since by definition, $y_i = x_i$ for each $i \in T$, this proves the theorem.

Let $S \in \mathcal{S}$. Observe that if $S \subseteq C_j\backslash\{j\}$ for some $j \in M$, then $\sum_{i\in S} y_i = \sum_{i\in S} m_i^{\sigma^j}(v) \geq v(S)$. If $S \subseteq T$, then $\sum_{i\in S} y_i = \sum_{i\in S} x_i \geq v(S)$. Finally, if $j_p \in S$, and $j_q \notin S$ for each $q \in \{p, \ldots, |M|\}$, then $y_{j_p} = \max\{v(U) - \sum_{k\in U\backslash\{j_p\}} y_k : U \subseteq T \cup \bigcup_{l=1}^{p} C_{j_l}, j_p \in U, U \in \mathcal{S}\}$ implies $\sum_{i\in S} y_i \geq v(S)$. We conclude that $\sum_{i\in S} y_i \geq v(S)$ for each $S \in \mathcal{S}$ and therefore that $y \in U(v)$. It remains to show that $\sum_{i\in N} y_i \leq v(N)$.

For each $p \in \{1, \ldots, |M|\}$, let $S^*_{j_p} \in \operatorname{argmax}\{v(U) - \sum_{k\in U\backslash\{j_p\}} y_k : U \subseteq T \cup \bigcup_{l=1}^{p} C_{j_l}, j_p \in U, U \in \mathcal{S}\}$ be maximal with respect to inclusion. To be more precise, if $R \in \operatorname{argmax}\{v(U) - \sum_{k\in U\backslash\{j_p\}} y_k : U \subseteq T \cup \bigcup_{l=1}^{p} C_{j_l}, j_p \in U, U \in \mathcal{S}\}$, and $R \neq S^*_{j_p}$, then $S^*_{j_p} \not\subseteq R$. Note that the following three properties, (P1), (P2) and (P3), are satisfied.

(P1) $S^*_{j_p} \cap S^*_{j_q} \neq \emptyset$ or $S^*_{j_p} \cup S^*_{j_q} \notin \mathcal{S}$ for each $p, q \in \{1, \ldots, |M|\}$ with $p < q$.

Indeed, suppose that $S^*_{j_p}$ and $S^*_{j_q}$ are such that $S^*_{j_p} \cap S^*_{j_q} = \emptyset$ and $S^*_{j_p} \cup S^*_{j_q} \in \mathcal{S}$. By definition of $S^*_{j_p}$, $\sum_{k\in S^*_{j_p}} y_k = v(S^*_{j_p})$. Now observe that

$$\begin{aligned} v(S^*_{j_q}) - \sum_{k\in S^*_{j_q}\backslash\{j_q\}} y_k &= v(S^*_{j_q}) - \sum_{k\in S^*_{j_q}\backslash\{j_q\}} y_k + v(S^*_{j_p}) - \sum_{k\in S^*_{j_p}} y_k \\ &\leq v(S^*_{j_p} \cup S^*_{j_q}) - \sum_{k\in (S^*_{j_p}\cup S^*_{j_q})\backslash\{j_q\}} y_k. \end{aligned}$$

The inequality follows from superadditivity. We conclude that $S^*_{j_p} \cup S^*_{j_q} \in \operatorname{argmax}\{v(U) - \sum_{k\in U\backslash\{j_q\}} y_k : U \subseteq T \cup \bigcup_{l=1}^{q} C_{j_l}, j_q \in U, U \in \mathcal{S}\}$. This contradicts the maximality of $S^*_{j_q}$.

(P2) $C_{j_p} \subseteq S^*_{j_p}$ for each $p \in \{1, \ldots, |M|\}$.

Indeed, if $C_{j_p} \nsubseteq S^*_{j_p}$, then $C_{j_p} \backslash S^*_{j_p} \neq \emptyset$. Let $W \subseteq C_{j_p} \backslash S^*_{j_p}$ be maximally connected. By definition of y and σ_{j_p}, $\sum_{i \in W} y_i = \sum_{i \in W} m_i^{\sigma_{j_p}}(v) = v(W)$. Hence, from superadditivity we obtain that $W \cup S^*_{j_p} \in \text{argmax}\{v(U) - \sum_{k \in U \backslash \{j_p\}} y_k : U \subseteq T \cup \bigcup_{l=1}^p C_{j_l}, j_p \in U, U \in \mathcal{S}\}$. This contradicts the maximality of $S^*_{j_p}$.

(P3) $T \cap S^*_{j_p} \neq \emptyset$ for each $p \in \{1, \ldots, |M|\}$.

Indeed, suppose $T \cap S^*_{j_p} = \emptyset$. Since $j_p \in S^*_{j_p}$, it follows that $T \cup S^*_{j_p} \in \mathcal{S}$. Hence, using $\sum_{i \in T} y_i = v(T)$ and superadditivity, we have $T \cup S^*_{j_p} \in \text{argmax}\{v(U) - \sum_{k \in U \backslash \{j_p\}} y_k : U \subseteq T \cup \bigcup_{l=1}^p C_{j_l}, j_p \in U, U \in \mathcal{S}\}$. This contradicts the maximality of $S^*_{j_p}$.

Now observe that $\bigcup_{S \in (\{T\} \cup \{S^*_j : j \in M\})} S = N$. From (P1) and (P3) it follows that if $A, B \in \{T\} \cup \{S^*_j : j \in M\}$ with $A \cap B = \emptyset$, then $A \cup B \notin \mathcal{S}$. By applying Lemma 3.3.2, with $\mathcal{V} = \{T\} \cup \{S^*_j : j \in M\}$, we conclude that $\{T\} \cup \{S^*_j : j \in M\}$ contains a covering family as a subset. Let $\mathcal{T} \subseteq \{T\} \cup \{S^*_j : j \in M\}$ be a covering family. From (P2) we deduce that each S^*_j, $j \in M$, contains at least one leaf. This implies that if T contains a leaf, or if $T \notin \mathcal{T}$, then $\mathcal{T}$ is a basic covering family without a central coalition. If $T \in \mathcal{T}$, and T does not contain a leaf, then $\mathcal{T}$ is a basic covering family with an essential central coalition. In any case, the associated covering family inequality is satisfied by assumption. We conclude

$$\begin{aligned}
\sum_{i \in N} y_i &= \sum_{S \in \mathcal{T}} \sum_{i \in S} y_i - \sum_{i \in N} \lambda_i(\mathcal{T}) y_i \\
&= \sum_{S \in \mathcal{T}} v(S) - \sum_{i \in N} \lambda_i(\mathcal{T}) y_i \\
&\leq v(N) + \max\{\sum_{S \in \mathcal{S}} u_S v(S) : u \in B(\mathcal{T})\} - \sum_{i \in N} \lambda_i(\mathcal{T}) y_i \\
&= v(N) + \max\{\sum_{S \in \mathcal{S}} u_S v(S) : u \in B(\mathcal{T})\} - \sum_{i \in W(\mathcal{T})} \lambda_i(\mathcal{T}) y_i \\
&\leq v(N).
\end{aligned}$$

The first equality is satisfied by definition of $\lambda(\mathcal{T})$. The second equality holds because $\sum_{i\in S} y_i = v(S)$ for each $S \in \{T\} \cup \{S_j^* : j \in M\}$. The first inequality is satisfied because of our assumption that the basic covering family inequality associated with $\mathcal{T}$ is satisfied. The third equality holds because $\lambda_i(\mathcal{T}) = 0$ for each $i \notin W(\mathcal{T})$. Finally, the last inequality holds because

$$\begin{aligned}
\sum_{i\in W(\mathcal{T})} \lambda_i(\mathcal{T}) y_i \quad &\geq \quad \min\{\sum_{i\in W(\mathcal{T})} \lambda_i(\mathcal{T}) z_i : \sum_{i\in S} z_i \geq v(S) \text{ for all } S \in \mathcal{S}\} \\
&\geq \quad \min\{\sum_{i\in W(\mathcal{T})} \lambda_i(\mathcal{T}) z_i : \sum_{i\in S} z_i \geq v(S) \\
&\qquad\qquad \text{for all } S \in \mathcal{S}(W(\mathcal{T}))\} \\
&= \quad \max\{\sum_{S\in\mathcal{S}} u_S v(S) : u \in B(\mathcal{T})\}.
\end{aligned}$$

We conclude that $\sum_{i\in N} y_i \leq v(N)$, and therefore $y \in C(v)$. $\square$

In the following example we illustrate the proof of Theorem 3.4.3.

Example 3.4.5 Let $G = (N, E)$ be the tree depicted in Figure 3.2 on page 54. Let the worths of the essential coalitions of the tree-component additive game $v \in TU^N$ be given by $v(\{i\}) = 0$ for each $i \in N$ and

$$v(S) = \begin{cases} 1, & \text{if } S = \{3,4\}, \{4,5\}; \\ 2, & \text{if } S = \{1,3,4\}, \{2,3,4\}, \{4,5,6\}, \{4,5,7\}; \\ 6, & \text{if } S = N. \end{cases}$$

The worths of the inessential coalitions are determined by superadditivity. In particular, if S is inessential, then there is a partition P of S such that $\sum_{T\in P} v(T) \geq v(S)$. By superadditivity it also follows that $\sum_{T\in P} v(T) \leq v(S)$ for each partition P of S. So if S is inessential, then $v(S) = \max \{\sum_{T\in P} v(T) : P \text{ is a partition of } S\}$.

It is straightforward to verify that (N, v) satisfies the condition of Theorem 3.4.3. Indeed, since the value of $v(N)$ is relatively high, the only basic covering family whose corresponding inequality might be violated is $\{\{1,3,4\}, \{2,3,4\}, \{4,5,6\}, \{4,5,7\}\}$. However, the corresponding inequality is satisfied with equality in this case.

Let $T = \{1,3,4\}$ and $(x_1, x_3, x_4) = (\frac{1}{2}, 1, \frac{1}{2}) \in C(v_T)$. Note that T is essential. We will extend x to a core element using the proof of Theorem 3.4.3.

Observe that $M = \{2,5\}$, $C_2 = \{2\}$ and $C_5 = \{5,6,7\}$. Let $\sigma_2 = (2)$ and $\sigma_5 = (6,7,5)$. Clearly, $N\backslash[\sigma^j(l), \sigma^j]$ is connected for each $j = 2,5$ and $l \in \{1,\ldots,|C_j|\}$. Let $y_1 = x_1 = \frac{1}{2}$, $y_3 = x_3 = 1$ and $y_4 = x_4 = \frac{1}{2}$. Furthermore,

$$\begin{aligned} y_2 &= \max\{v(U) - \sum_{j\in U\backslash\{2\}} y_j : U \subseteq \{1,2,3,4\}, 2 \in U, U \in \mathcal{S}\} \\ &= v(\{2,3,4\}) - y_3 - y_4 \\ &= \frac{1}{2}. \end{aligned}$$

Finally, $y_6 = m_6^{\sigma^5}(v) = v(\{6\}) = 0$, $y_7 = m_7^{\sigma^5}(v) = v(\{6,7\}) - v(\{6\}) = 0$ and

$$\begin{aligned} y_5 &= \max\{v(U) - \sum_{j\in U\backslash\{5\}} y_j : U \subseteq N, 5 \in U, U \in \mathcal{S}\} \\ &= v(N) - y_1 - y_2 - y_3 - y_4 - y_6 - y_7 \\ &= 3\frac{1}{2}. \end{aligned}$$

Observe that $y = (\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, 3\frac{1}{2}, 0, 0) \in C(v)$ is an extension of x. $\diamond$

Note that by virtue of Theorem 3.4.2 the condition of Theorem 3.4.3 is also sufficient for core stability. We conclude this section with several examples. The first example shows that the condition of Theorem 3.4.3 is not necessary for essential extendibility. Secondly, an example of a tree-component additive game is provided that has a stable core, although this game is not essential extendible. Hence, the basic covering family inequalities are not necessary for essential extendibility, and essential extendibility is, on its turn, not necessary for core stability. In the upcoming section we show that in chain-component additive games essential extendibility is equivalent to core stability, and we characterise these concepts in terms of basic covering family inequalities.

Example 3.4.6 Let $G = (N, E)$ be the tree depicted in Figure 3.2 on page 54. Let $v \in TU^N$ be the tree-component additive game with respect to G, where the worths of the essential coalitions are given by $v(\{i\}) = 0$ for each $i \in N$, $v(\{3,4\}) = v(\{4,5\}) = 1$, $v(\{1,3,4\}) = v(\{2,3,4\}) = v(\{4,5,6\}) = v(\{4,5,7\}) = 2$ and $v(N) = 5$. The worths of the inessential coalitions are determined by superadditivity (as explained in Example 3.4.5).

First note that the inequality associated to the basic covering family $\{\{1,3,4\},\{2,3,4\},\{4,5,6\},\{4,5,7\}\}$ is violated. Indeed, it is straightforward to show that $\max\{\sum_{S\in\mathcal{S}} y_S v(S) : y \in B(\mathcal{T})\} = v(\{4\}) + v(\{3,4\}) + v(\{4,5\}) = 2$. This yields, $v(\{1,3,4\})+v(\{2,3,4\})+v(\{4,5,6\})+v(\{4,5,7\}) = 8 > 7 = v(N) + \max\{\sum_{S\in\mathcal{S}} y_S v(S) : y \in B(\mathcal{T})\}$. However, we will show that (N, v) is essential extendible. Let $S \subseteq N$ be essential.

If S is a singleton, i.e. $S = \{i\}$ for some $i \in N$, then each $x \in C(v_S)$ can easily be extended to a core element. Just observe that $x = 0$ for each $x \in C(v_S)$, and note that $(0,0,0,5,0,0,0)$ and $(0,0,2\frac{1}{2},0,2\frac{1}{2},0,0)$ are core elements.

Now suppose $S = \{3,4\}$ and let $(x_3, x_4) \in C(v_S)$. So $x_3 + x_4 = 1$ and $x_3, x_4 \geq 0$. This implies that x can be extended to $y = (1,1,x_3,x_4,2,0,0) \in C(v)$. Note that by using similar arguments it is straightforward to extend each core element of $C(v_S)$, if $S = \{4,5\}$.

Finally, let $S = \{1,3,4\}$ and let $(x_1, x_3, x_4) \in C(v_S)$. So $x_1+x_3+x_4 = 2$, $x_3 + x_4 \geq 1$ and $x_1, x_3, x_4 \geq 0$. We conclude that $y = (x_1, 1, x_3, x_4, 2, 0, 0)$ is a core element that extends x. Again, using similar arguments, it is straightforward to extend each core element of $C(v_S)$ if $S = \{2,3,4\}, \{4,5,6\}$ or $\{4,5,7\}$. $\diamond$

Example 3.4.7 Let $G = (N, E)$ be the tree depicted in Figure 3.4. Let $v \in TU^N$ be the tree-component additive game with respect to G, where the worths of the essential coalitions are given by $v(\{i\}) = 0$ for each $i \in N$, $v(\{1,2\}) = v(\{3,4\}) = v(\{4,6\}) = v(\{7,8\}) = v(\{2,4,5,7\}) = 1$ and $v(N) = 4$. The worths of the inessential coalitions now follow from superadditivity (as explained in Example 3.4.5).

First we show that (N, v) is not essential extendible. Observe that

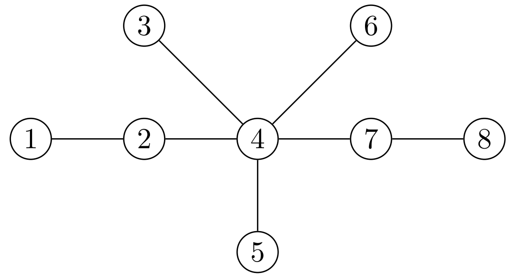

Figure 3.4: A tree $G = (N, E)$.

$\{2,4,5,7\}$ is essential. However, $(x_2, x_4, x_5, x_7) = (0,0,1,0) \in C(v_{\{2,4,5,7\}})$ cannot be extended to a core element. Indeed, let y be such that $y_i = x_i = 0$ for each $i \in \{2,4,7\}$, $y_5 = x_5 = 1$, $y_1 + y_2 \geq v(\{1,2\})$, $y_3 + y_4 \geq v(\{3,4\})$, $y_4 + y_6 \geq v(\{4,6\})$ and $y_7 + y_8 \geq v(\{7,8\})$. This implies $y_1 \geq 1$, $y_3 \geq 1$, $y_6 \geq 1$ and $y_8 \geq 1$. Therefore, $\sum_{i \in N} y_i \geq 5 > 4 = v(N)$. We conclude that $y \notin C(v)$.

Now we show that $C(v)$ is stable. Let $x \in I(v) \backslash C(v)$ and let $S \subseteq N$ be an essential coalition with $\sum_{i \in S} x_i < v(S)$.

First suppose that $S = \{1,2\}$. Then x is dominated, via coalition $\{1,2\}$, by the core element

$$y = (x_1 + \frac{v(\{1,2\}) - x_1 - x_2}{2}, x_2 + \frac{v(\{1,2\}) - x_1 - x_2}{2}, 0, 2, 0, 0, 1, 0).$$

Secondly, suppose that $S = \{3,4\}$. Then x is dominated via $\{3,4\}$ by the core element

$$y = (0, 1, x_3 + \frac{v(\{3,4\}) - x_3 - x_4}{2}, x_4 + \frac{v(\{3,4\}) - x_3 - x_4}{2}, 0, 1, 1, 0).$$

By using similar arguments it is straightforward to obtain core elements that dominate x if $S = \{4,6\}, \{7,8\}$.

Finally, suppose that $S = \{2,4,5,7\}$. We may assume that $x_1 + x_2 \geq 1$, $x_3 + x_4 \geq 1$, $x_4 + x_6 \geq 1$ and $x_7 + x_8 \geq 1$, since if one of these inequalities is not satisfied, then x can easily be dominated, as we have seen above. Adding the four inequalities yields $\sum_{i \in N} x_i + x_4 - x_5 \geq 4$. Together with $\sum_{i \in N} x_i = 4$ we obtain $x_4 \geq x_5$. Now define $\varepsilon = v(\{2,4,5,7\}) - x_2 - x_4 - x_5 - x_7$ and let y be such that $y_i = x_i + \frac{\varepsilon}{4}$ for $i \in \{2,4,5,7\}$, $y_1 = 1 - y_2$, $y_3 = 1 - y_4$, $y_8 = 1 - y_7$ and $y_6 = v(N) - \sum_{j \in N \setminus \{6\}} y_j = 4 - \sum_{j \in N \setminus \{6\}} y_j$. Clearly, y dominates x via $\{2,4,5,7\}$. It remains to show that $y \in C(v)$.

First observe that by definition of y_6, $\sum_{j \in N} y_j = 4 = v(N)$. Secondly, observe that $x_2 + x_4 + x_5 + x_7 < 1$ and $x_4, x_5, x_7 \geq 0$ imply that $x_2 < 1$. Therefore also $y_2 < 1$. As a result, $y_1 > 0$. Similarly, it follows that $y_3 > 0$, $y_8 > 0$ and $y_6 = 4 - \sum_{i \in N \setminus \{6\}} y_i = 1 - y_5 > 0$. Because $y_1 + y_2 \geq 1$, $y_3 + y_4 \geq 1$, $y_7 + y_8 \geq 1$ and $y_2 + y_4 + y_5 + y_7 \geq 1$, it remains to show that $y_4 + y_6 \geq 1$. Note that

$$\begin{aligned} y_4 + y_6 &= 4 - y_1 - y_2 - y_3 - y_5 - y_7 - y_8 \\ &= 2 - y_3 - y_5 \\ &\geq 2 - y_3 - y_4 \\ &= 1. \end{aligned}$$

The inequality is satisfied since $x_4 \geq x_5$ and therefore $y_4 \geq y_5$. We conclude that $y \in C(v)$ dominates x via $\{2,4,5,7\}$. $\diamond$

The last example of this section proves that exactness is not a sufficient condition for core stability in tree-component additive games, although in the next section it is proved that exactness is sufficient for core stability on the class of chain-component additive games.

Example 3.4.8 Let $G = (N, E)$ be the tree depicted in Figure 3.1 on page 52 and $v \in TU^N$ be the tree-component additive game of Example 3.3.3. In Example 3.3.4 we have shown that (N, v) is exact. We claim that $x = (\frac{1}{2}, 2, 0, 1, \frac{1}{2}) \in I(v) \setminus C(v)$ cannot be dominated by a core element. Indeed, suppose that $y \in C(v)$ dominates x. Then this domination must occur via coalition $\{1,3,4\}$ or coalition $\{3,4,5\}$, since these are the only dissatisfied coalitions at x.

First suppose that the domination occurs via $\{1,3,4\}$. Then $y_1 + y_3 + y_4 = 2$, $y_1 > x_1 = \frac{1}{2}$, $y_3 > x_3 = 0$ and $y_4 > x_4 = 1$. Hence, $y_3 = 2 - y_1 - y_4 < \frac{1}{2}$ and $y_3 + y_4 = 2 - y_1 < 1\frac{1}{2}$. Since $y \in C(v)$, $y_2 \geq v(\{2,3\}) - y_3 > 2 - \frac{1}{2} = 1\frac{1}{2}$ and $y_5 \geq v(\{3,4,5\}) - y_3 - y_4 > 2 - 1\frac{1}{2} = \frac{1}{2}$. We conclude that $\sum_{i \in N} y_i > 4$, which contradicts $y \in C(v)$.

Finally, suppose that the domination occurs via $\{3,4,5\}$. Then $y_3 + y_4 + y_5 = 2$, $y_3 > x_3 = 0$, $y_4 > x_4 = 1$ and $y_5 > x_5 = \frac{1}{2}$. Therefore, $y_3 = 2 - y_4 - y_5 < \frac{1}{2}$ and $y_3 + y_4 = 2 - y_5 < 1\frac{1}{2}$. From $y \in C(v)$ we deduce that $y_2 \geq v(\{2,3\}) - y_3 > 1\frac{1}{2}$ and $y_1 \geq v(\{1,3,4\}) - y_3 - y_4 > \frac{1}{2}$. So, $\sum_{i \in N} y_i > 4$, and therefore $y \notin C(v)$. $\diamond$

3.5 Chain-component additive games

In this section we study largeness, exactness, essential extendibility and core stability on the class of chain-component additive games. We show the equivalence between largeness and exactness. Furthermore we prove that essential extendibility is equivalent to core stability and we characterise both concepts in terms of polynomially many linear equalities and inequalities. We begin this section with a description of covering families on chains.

Let $G = (N, E)$ be a chain. We will, without loss of generality, assume throughout this section that $\{i, i+1\} \in E$ for each $i \in N \backslash \{|N|\}$. For convenience, we now redefine the concept of covering families in a slightly different way. An ordered set $\{T_1, \ldots, T_m\} \subseteq \mathcal{S}$ is an *m-covering family* if

(B1) $\bigcup_{i=1}^{m} T_i = N$;

(B2) $\bigcup_{i=1, i \neq j}^{m} T_i \neq N$ for each $j \in \{1, \ldots, m\}$;

(B3) $T_i \cap T_{i+1} \neq \emptyset$ for all $i \in \{1, \ldots, m-1\}$.

Requirements (B1) and (B2) are similar to requirements (A1) and (A2) for covering families on trees. The third requirement, (B3), states that two subsequent elements of an m-covering family should not be disjoint. Observe that if an ordered set $\mathcal{T} \subseteq \mathcal{S}$ satisfies (B1), (B2) and (B3), then it necessarily also satisfies (A1), (A2) and (A3). Similarly, if an unordered set $\mathcal{T} \subseteq \mathcal{S}$ satisfies (A1), (A2) and (A3), then it satisfies (B1) and (B2). Furthermore, the elements of $\mathcal{T}$ can be ordered such that (B3) is satisfied as well.

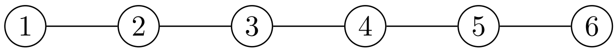

Figure 3.5: A chain $G = (N, E)$.

Example 3.5.1 Let $G = (N, E)$ be the chain depicted in Figure 3.5. Then $\{\{1, 2, 3\}, \{2, 3, 4, 5\}, \{5, 6\}\}$ forms a 3-covering family. But for instance, $\{\{1, 2, 3\}, \{2, 3, 4\}, \{4, 5, 6\}\}$ is not a 3-covering family, since (B2) is violated. $\diamond$

Observe that an m-covering family could equivalently be described by the alternating sequence of $2m-1$ non-empty blocks of consecutive players who are covered by exactly one or exactly two family-member coalitions. It follows that in an n-player chain-component additive game the number of different m-covering families is $\binom{n-1}{2m-2}$, provided, of course, that $2m-1 \leq n$.

Let $G = (N, E)$ be a chain, and let $v \in TU^N$ be chain-component additive with respect to G. For each m-covering family $\mathcal{T} = \{T_1, \ldots, T_m\}$ the associated covering family inequality (3.1) boils down to

$$\sum_{i=1}^{m} v(T_i) \leq v(N) + \sum_{i=1}^{m-1} v(T_i \cap T_{i+1}).$$

Observe that $\max\{\sum_{S \in \mathcal{S}} y_S v(S) : y \in B(\mathcal{T})\} = \sum_{i=1}^{m-1} v(T_i \cap T_{i+1})$ because $\lambda_i(\mathcal{T}) \in \{0, 1\}$ for each $i \in N$ and because of superadditivity. The following theorem characterises largeness, extendibility and exactness in terms of covering family inequalities.

Theorem 3.5.1 Let $G = (N, E)$ be a chain, and let $v \in TU^N$ be a chain-component additive game with respect to G. The following statements are equivalent:

1. Each covering family inequality is satisfied;

2. $C(v)$ is large;

3. (N, v) is extendible;

4. (N, v) is exact.

Proof: The equivalence between 1 and 2 is proved in Theorem 3.3.1. It is shown in Kikuta and Shapley (1986) that 2 $\Rightarrow$ 3, and 3 $\Rightarrow$ 4 follows from Biswas, Parthasarathy, Potters, and Voorneveld (1999) since chain-component additive games have non-empty cores, as well as all subgames of chain-component additive games. It remains to show that 4 $\Rightarrow$ 1.

Assume that (N, v) is exact. Let $\mathcal{T} = \{T_1, \ldots, T_m\}$ be an m-covering family. Since (N, v) is exact it follows from Theorem 3.3.3 that for each $T \subseteq N$, and each $y : \mathcal{S} \to \mathbb{R}_+$ with $\sum_{S\in\mathcal{S}\setminus\{N\}} y_S e(S) = y_N e(N) + e(T)$,

$$\sum_{S\in\mathcal{S}\setminus\{N\}} y_S v(S) \leq y_N v(N) + v(T). \tag{3.10}$$

Now let $T = \bigcup_{i=1}^{m-1}(T_i \cap T_{i+1})$ and let $y : \mathcal{S} \to \mathbb{R}_+$ be given by $y_S = 1$ if $S \in \mathcal{T} \cup \{N\}$ and $y_S = 0$ otherwise. Clearly, $\sum_{S\in\mathcal{S}\setminus\{N\}} y_S e(S) = y_N e(N) + e(T)$. It follows that

$$\begin{aligned}
\sum_{i=1}^{m} v(T_i) &= \sum_{S\in\mathcal{S}\setminus\{N\}} y_S v(S) \\
&\leq y_N v(N) + v(T) \\
&= v(N) + \sum_{i=1}^{m-1} v(T_i \cap T_{i+1}).
\end{aligned}$$

The inequality follows from (3.10). The last equality is satisfied by definition of T, and the chain-component additivity of (N, v). $\square$

Observe that for an n-player chain-component additive game there are exactly $2^{n-2} - 1$ covering families. Indeed, for each $S \subseteq N\setminus\{1, n\}$ with $S \neq \emptyset$ there is a covering family $\mathcal{T} = \{T_1, \ldots, T_m\}$ with $\bigcup_{i=1}^{m-1}(T_i \cap T_{i+1}) = S$. Hence, our characterisation requires the checking of $2^{n-2} - 1$ linear inequalities. Also note that, since largeness of the core is a sufficient condition for essential extendibility and core stability, exactness is sufficient for essential extendibility and core stability on the class of chain-component additive games as well.

In the final part of this section we characterise essential extendibility and core stability in chain-component additive games. Of course, Theorem 3.4.3 provides a sufficient condition for essential extendibility and core stability on the class of tree-component additive games. We show that for chain-component additive games these conditions are necessary as well. First observe that if $G = (N, E)$ is a chain, then a basic covering family without a central coalition is a 2-covering family, and a basic covering family with a central coalition is just a 3-covering family.

Before we show the characterisation of essential extendibility and core stability, we first apply Theorem 3.4.1 to chain-component additive games. Then we recall a variant of Farkas' Lemma, which was first proved in Haar (1918). This lemma is also recorded in e.g. Schrijver (1986).

Theorem 3.5.2 Let $G = (N, E)$ be a chain and let $v \in TU^N$ be a chain-component additive game with respect to G. If $\sigma \in \Pi(N)$ is connected with respect to G, then $m^\sigma(v) \in C(v)$.

Note that from Theorem 3.5.2 it follows that $m^\sigma(v), m^\tau(v) \in C(v)$ with $\sigma, \tau \in \Pi(N)$ such that $\sigma(i) = i$, $\tau(i) = |N| + 1 - i$ for all $i \in \{1, \ldots, |N|\}$. This is also indirectly proved in Curiel, Potters, Rajendra Prasad, Tijs, and Veltman (1994).

Lemma 3.5.1 (Haar (1918)) Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ such that $P = \{x \in \mathbb{R}^n : Ax \leq b\} \neq \emptyset$. Let $c \in \mathbb{R}^n$ and $\delta \in \mathbb{R}$. Then $cx \leq \delta$ for all $x \in P$ if and only if there exists a $y \in \mathbb{R}^m_+$ with $yA = c$ and $yb \leq \delta$.

Theorem 3.5.3 Let $G = (N, E)$ be a chain, and let $v \in TU^N$ be tree-component additive with respect to G. The following statements are equivalent:

1. Each 2-covering family inequality is satisfied. For each 3-covering family $\{T_1, T_2, T_3\}$ with essential T_2, the corresponding inequality is satisfied;

2. (N, v) is essential extendible;

3. $C(v)$ is stable.

Proof: We only show $3 \Rightarrow 1$ since $1 \Rightarrow 2$ follows from Theorem 3.4.3, and $2 \Rightarrow 3$ from Theorem 3.4.2. We first show that the inequalities corresponding to 2-covering families are necessary. We then proceed with the necessity of the condition involving 3-covering families.

Suppose that the inequality corresponding to the 2-covering family $\{T_1, T_2\}$ is violated. In other words, suppose that $v(T_1) + v(T_2) > v(N) + v(T_1 \cap T_2)$. We will show that the core is not stable by constructing a non-core imputation that is not be dominated by any core element.

Let t^* be such that $T_1 = \{1, \ldots, t^*\}$, and consider the order $\sigma \in \Pi(N)$ with $\sigma(i) = t^*+1-i$ for each $i \in \{1, \ldots, t^*\}$ and $\sigma(i) = |N|+t^*+1-i$ for each $i \in \{t^*+1, \ldots, |N|\}$. Observe $\sum_{i \in T_2} m_i^\sigma(v) = v(N) - v(T_1) + v(T_1 \cap T_2) < v(T_2)$, where the inequality follows by assumption. Thus, $m^\sigma(v) \notin C(v)$. Furthermore, from superadditivity we conclude that $m^\sigma(v) \in I(v)$.

Now we will show that $\sum_{i \in S} m_i^\sigma(v) \geq v(S)$ for all $S \in \mathcal{S}$ with $t^*+1 \notin S$. This implies that $\sum_{i \in S} m_i^\sigma(v) \geq v(S)$ for all $S \subseteq N \backslash \{t^*+1\}$, and therefore that $m^\sigma(v)$ can only be dominated by coalitions containing player t^*+1.

Let $S \in \mathcal{S}$ be such that $t^*+1 \notin S$, and let $T \in \mathcal{C}(N \backslash \{t^*+1\})$ be such that $S \subseteq T$. Consider the subgame (T, v_T), and let $\sigma_T \in \Pi(T)$ be σ restricted to T, i.e. σ_T is such that for all $i, j \in T$, $\sigma_T^{-1}(i) < \sigma_T^{-1}(j)$ if and only if $\sigma^{-1}(i) < \sigma^{-1}(j)$. From the chain-component additivity of (N, v) it follows that $m_i^\sigma(v) = m_i^{\sigma_T}(v_T)$ for all $i \in T$. Therefore $\sum_{i \in S} m_i^\sigma(v) = \sum_{i \in S} m_i^{\sigma_T}(v_T) \geq v_T(S) = v(S)$, where the inequality is satisfied because of Theorem 3.5.2 and the fact that σ_T is connected with respect to (T, E_T).

We concluded that $m^\sigma(v)$ can only be dominated via coalitions that contain player t^*+1. However, at $m^\sigma(v)$ player t^*+1 receives a payoff of $m_{t^*+1}^\sigma(v) = v(N) - v(\{1, \ldots, t^*\}) - v(\{t^*+2, \ldots, n\}) = v(N) - v(N \backslash \{t^*+1\})$. But for any $x \in C(v)$, $x_{t^*+1} \leq v(N) - v(N \backslash \{t^*+1\})$. We therefore conclude that $m^\sigma(v)$ cannot be dominated by a core element via a coalition containing player t^*+1. This implies that $m^\sigma(v)$ cannot be dominated by any core element. So the core is not stable. Consequently, the inequalities arising from 2-covering families are necessary for core stability.

We now show that the condition for 3-covering families is necessary. Assume that the inequalities corresponding to 2-covering families are satisfied. Furthermore suppose that for the 3-covering family $\{T_1, T_2, T_3\}$ the corresponding condition is violated, i.e. suppose that $v(T_1) + v(T_2) + v(T_3) > v(N) + v(T_1 \cap T_2) + v(T_2 \cap T_3)$ while T_2 is essential. Again, we show that the core is not stable by showing the existence of a non-core imputation that can not be dominated by any core element. Before we actually start the proof, we first introduce some notation.

Define $T^* = N \backslash (T_1 \cup T_3)$ and $\mathcal{T} = \{T \in \mathcal{S} : T^* \not\subseteq T\}$. So $\mathcal{T}$ is the set of connected coalitions not containing T^*. Define P by

$$\begin{aligned} P \ &= \{x \in \mathbb{R}^N : \ \sum_{i \in T} x_i \geq v(T) \text{ for all } T \in \mathcal{T}, \sum_{i \in N} x_i \leq v(N), \\ &\qquad \sum_{i \in T^*} x_i \geq v(N) - v(T_1) - v(T_3)\}. \end{aligned}$$

Firstly we show that $P \neq \emptyset$ and that $P \subseteq I(v)$. Secondly we show, by applying Lemma 3.5.1, the existence of an $x \in P \backslash C(v)$ that cannot be dominated by any core element. This implies the necessity of the condition involving 3-covering families.

Let $T_1 = \{1, \ldots, t_1\}$ and consider $\sigma \in \Pi(N)$ with $\sigma(i) = i$ for all $i \in \{1, \ldots, t_1\}$ and $\sigma(i) = |N| + t_1 + 1 - i$ for all $i \in \{t_1 + 1, \ldots, |N|\}$. From Theorem 3.5.2 it follows that $m^\sigma(v) \in C(v)$. Consequently, we have $\sum_{i \in T} m_i^\sigma(v) \geq v(T)$ for all $T \in \mathcal{T}$, and that $\sum_{i \in N} m_i^\sigma(v) = v(N)$. Furthermore, observe that $\sum\limits_{i \in T^*} m_i^\sigma(v) = v(N) - v(T_1) - v(T_3)$. We conclude that $m^\sigma(v) \in P$, and thus that $P \neq \emptyset$.

Next we show that $P \subseteq I(v)$. Let $x \in P$. We need to show that $\sum_{i \in N} x_i = v(N)$ and $x_i \geq v(\{i\})$ for all $i \in N$. Since $T_1, T_3 \in \mathcal{T}$, $\sum_{i \in T_1} x_i \geq v(T_1)$ and $\sum_{i \in T_3} x_i \geq v(T_3)$. Because $\sum_{i \in T^*} x_i \geq v(N) - v(T_1) - v(T_3)$ it follows that $\sum_{i \in N} x_i \geq v(N)$. By definition of P, $\sum_{i \in N} x_i \leq v(N)$. So we conclude that $\sum_{i \in N} x_i = v(N)$. Because $\{i\} \in \mathcal{T}$ for each $i \notin T^*$, $x_i \geq v(\{i\})$ for all $i \notin T^*$. If $|T^*| > 1$, then $\{i\} \in \mathcal{T}$ for all $i \in T^*$. So in this case we have that $x_i \geq v(\{i\})$ for all $i \in N$. If $|T^*| = 1$, then $T^* = \{i\}$ for some $i \in N$, and consequently we have that $\{i\} \notin \mathcal{T}$. However, observe that $x_i = \sum_{j \in T^*} x_j \geq v(N) - v(T_1) - v(T_3) \geq v(T^*) = v(\{i\})$, where the first inequality follows since $x \in P$ and the second by superadditivity. So if $|T^*| = 1$, then $x_i \geq v(\{i\})$ for all $i \in N$. We conclude that $P \subseteq I(v)$.

It remains to show the existence of an $x \in P \backslash C(v)$ that cannot be dominated by any core element. In order to do so, we define a matrix A and a vector b such that $x \in P$ if and only if $Ax \leq b$. So for each $T \in \mathcal{T} \cup \{T^*\}$ there is a corresponding row $-e(T)$ in A and for coalition N there is a row $e(N)$ in A. Similarly, $b_i = -v(T)$ if the i-th row in A is $-e(T)$, $T \in \mathcal{T}$. Furthermore, $b_i = -v(N) + v(T_1) + v(T_3)$ if the i-th row in A is $-e(T^*)$, and $b_i = v(N)$ if the i-th row in A is $e(N)$. Since P is non-empty, $Ax \leq b$ has a solution. So we can apply Lemma 3.5.1, with $c = -e(T_2)$ and $\delta = -v(T_2)$, to conclude that for all $x \in P$, $-e(T_2)x = -\sum_{i \in T_2} x_i \leq -v(T_2)$ if and only if there is a $y \geq 0$ with $yA = -e(T_2)$ and $yb \leq -v(T_2)$. However, we will show that for all $y \geq 0$ with $yA = -e(T_2)$, $yb > -v(T_2)$. This means there is an $x \in P$ with $-\sum_{i \in T_2} x_i > -v(T_2)$. Hence, $\sum_{i \in T_2} x_i < v(T_2)$ and therefore $x \notin C(v)$. By definition of P, x can only be dominated by coalitions containing T^*. But for every $y \in C(v)$ we have that $\sum_{j \in T_1} y_j \geq v(T_1)$, $\sum_{j \in T_3} y_j \geq v(T_3)$ and $\sum_{j \in N} y_j = v(N)$. Consequently, $\sum_{j \in T^*} y_j \leq v(N) - v(T_1) - v(T_3)$ for every $y \in C(v)$. That is, at x coalition T^* receives a payoff that is at least as much as its highest payoff at any core allocation. So x can not be dominated by a core element via a coalition that contains T^*. Consequently, the core is not stable. This implies that for core stability the conditions corresponding to 3-covering families are necessary.

It remains to show that for all $y \geq 0$ with $yA = -e(T_2)$, $yb > -v(T_2)$. For each $|\mathcal{T} \cup \{T^*, N\}|$ -dimensional vector $u \geq 0$ we write, with abuse of notation, u_S instead of u_i if the i-th row of A is the row corresponding to coalition S. Define $\mathcal{Y}(u) = \{S \in \mathcal{T} \cup \{T^*, N\} : u_S > 0\}$ as the set of coalitions that u assigns a positive weight to. Let $y \geq 0$ be such that $yA = -e(T_2)$. Instead of calculating yb directly, we first decompose y by using Lemmas 3.6.1, 3.6.4 and 3.6.5. These lemmas are stated and proved in Section 3.6. Then we derive the product of these decomposition vectors with b. This enables us to obtain a bound for yb.

According to Lemma 3.6.1 we can decompose y into $\sum_{k=1}^{a_1} \lambda_k u^k + r^1$, with $r^1 \geq 0$, $r^1 A = -e(T_2)$, $\mathcal{Y}(r^1) \backslash \{N\}$ contains no partition of N and for all $k \in \{1,\dots,a_1\}$, $\lambda_k > 0$ and u^k satisfies (A1). Observe that r^1 satisfies the conditions of Lemma 3.6.4. It follows that $r^1 = \sum_{k=1}^{a_2} \mu_k w^k + r^2$ with $r^2 \geq 0$, $r^2_N = 0$, $\sum_{k=1}^{a_2} \mu_k \leq 1$ and for all $k \in \{1,\dots,a_2\}$, $\mu_k > 0$ and w^k satisfies (A2). This implies, because $r^1 A = -e(T_2)$ and $w^k A = -e(T_2)$ for all $k \in \{1,\dots,a_2\}$, that $r^2 A = (1 - \sum_{k=1}^{a_2} \mu_k)(-e(T_2))$. Note that, since $0 \leq 1 - \sum_{k=1}^{a_2} \mu_k \leq 1$, it follows that r^2 satisfies the condition of Lemma 3.6.5. Therefore we can write $r^2 = \sum_{k=1}^{a_3} \nu_k z^k$ with $\sum_{k=1}^{a_3} \nu_k = 1 - \sum_{k=1}^{a_2} \mu_k$ and for all $k \in \{1,\dots,a_3\}$, $\nu_k > 0$ and z^k satisfies (A3). Concluding, we have $y = \sum_{k=1}^{a_1} \lambda_k u^k + \sum_{k=1}^{a_2} \mu_k w^k + \sum_{k=1}^{a_3} \nu_k z^k$ with $\sum_{k=1}^{a_2} \mu_k + \sum_{k=1}^{a_3} \nu_k = 1$.

Before we show that $yb > -v(T_2)$ we first find bounds for $u^k b$, $w^k b$ and $z^k b$. Let $k \in \{1,\dots,a_1\}$. First suppose that $T^* \notin \mathcal{Y}(u^k)$. Then

$$u^k b = \sum_{S \in \mathcal{Y}(u^k) \backslash \{N\}} [-v(S)] + v(N) \geq 0.$$

Here the inequality is satisfied because of superadditivity and because $\mathcal{Y}(u^k) \backslash \{N\}$ is a partition of N. Now suppose that $T^* \in \mathcal{Y}(u^k)$. Since $\mathcal{Y}(u^k) \backslash \{N\}$ is a partition of N it follows that $\mathcal{Y}(u^k) \backslash \{T^*, N\}$ consists of a partition A of T_1 and a partition B of T_3. It follows that

$$\begin{aligned} u^k b &= \sum_{S \in \mathcal{Y}(u^k) \backslash \{T^*, N\}} [-v(S)] + [v(T_1) + v(T_3) - v(N)] + v(N) \\ &= \sum_{S \in A} [-v(S)] + \sum_{S \in B} [-v(S)] + v(T_1) + v(T_3) \geq 0. \end{aligned}$$

The inequality holds because of superadditivity and because A is a partition of T_1 and B a partition of T_3. Concluding, for all $k \in \{1,\dots,a_1\}$,

$$u^k b \geq 0. \tag{3.11}$$

Now let $k \in \{1,\dots,a_2\}$. First suppose that $T^* \notin \mathcal{Y}(w^k)$. Then $\mathcal{Y}(w^k) = U_k \cup V_k \cup \{N\}$, where U_k is a partition of $T_1 \cup T_2$ and V_k a partition of $T_2 \cup T_3$ with $U_k \cap V_k = \emptyset$. Let $\bar{V}_k$ consist of those elements of V_k that are not a subset of T_2, i.e. $\bar{V}_k = \{T \in V_k : T \not\subseteq T_2\}$. Therefore

$$\begin{aligned}
w^k b &= \sum_{S \in \mathcal{Y}(w^k)\setminus\{N\}} [-v(S)] + v(N) \\
&= \sum_{S \in U_k} [-v(S)] + \sum_{S \in V_k} [-v(S)] + v(N) \\
&\geq \sum_{S \in U_k} [-v(S)] + \sum_{S \in V_k} [-v(S)] + v(T_1 \cup T_2) + v(\bigcup_{T \in \bar{V}_k} T) \\
&\quad -v(T_2 \cap (\bigcup_{T \in \bar{V}_k} T)) \\
&\geq \sum_{S \in V_k} [-v(S)] + v(\bigcup_{T \in \bar{V}_k} T) - v(T_2 \cap (\bigcup_{T \in \bar{V}_k} T)) \\
&\geq \sum_{S \in V_k} [-v(S)] + \sum_{T \in \bar{V}_k} v(T) - v(T_2 \cap (\bigcup_{T \in \bar{V}_k} T)) \\
&= \sum_{S \in V_k \setminus \bar{V}_k} [-v(S)] - v(T_2 \cap (\bigcup_{T \in \bar{V}_k} T)) \\
&\geq -v(\bigcup_{S \in V_k \setminus \bar{V}_k} S) - v(T_2 \cap (\bigcup_{T \in \bar{V}_k} T)) \\
&> -v(T_2).
\end{aligned}$$

We first explain the first inequality. According to Lemma 3.6.6 there is a $T \in \bar{V}_k$ with $T \cap T_2 \neq \emptyset$. This implies that $\{T_1 \cup T_2, \bigcup_{T \in \bar{V}_k} T\}$ forms a 2-covering family. Observe that because of Lemma 3.6.6, $(T_1 \cup T_2) \cap (\bigcup_{T \in \bar{V}_k} T) = T_2 \cap (\bigcup_{T \in \bar{V}_k} T)$. Since we have assumed that the inequalities corresponding to 2-covering families hold, the first inequality is satisfied. The second inequality holds because of superadditivity and because U_k is a partition of $T_1 \cup T_2$. The third and fourth inequalities are satisfied due to superadditivity. Finally we explain the last inequality. According to Lemma 3.6.6, $\bar{V}_k$ and $V_k \setminus \bar{V}_k$ are both non-empty, and there is a $T \in \bar{V}_k$ with $T \cap T_2 \neq \emptyset$. This means that $\{\bigcup_{S \in V_k \setminus \bar{V}_k} S, T_2 \cap (\bigcup_{T \in \bar{V}_k} T)\}$ forms a partition of T_2. Because of our assumption that T_2 is essential the last inequality is satisfied.

Now suppose that $T^* \in \mathcal{Y}(w^k)$. Since $U_k \cap V_k = \emptyset$, either $T^* \in U_k$ or $T^* \in V_k$. Without loss of generality assume that $T^* \in U_k$. Now observe, since U_k is a partition of $T_1 \cup T_2$, that $U_k \setminus \{T^*\}$ consists of a partition C of

T_1 and a partition D of $T_2 \cap T_3$. Therefore

$$\begin{aligned} w^k b &= \sum_{S \in \mathcal{Y}(w^k) \backslash \{T^*, N\}} [-v(S)] + [v(T_1) + v(T_3) - v(N)] + v(N) \\ &> \sum_{S \in \mathcal{Y}(w^k) \backslash \{T^*, N\}} [-v(S)] + [v(T_1 \cap T_2) + v(T_2 \cap T_3) - v(T_2)] + v(N) \\ &= \sum_{S \in C} [-v(S)] + \sum_{S \in D} [-v(S)] + \sum_{S \in V_k} [-v(S)] + v(T_1 \cap T_2) \\ &\quad + v(T_2 \cap T_3) - v(T_2) + v(N) \\ &\geq -v(T_1) - v(T_2 \cap T_3) - v(T_2 \cup T_3) + v(T_1 \cap T_2) + v(T_2 \cap T_3) \\ &\quad - v(T_2) + v(N) \\ &\geq -v(T_2). \end{aligned}$$

The first inequality holds since we have assumed that the 3-covering family inequality corresponding to $\{T_1, T_2, T_3\}$ is violated. The second inequality because of superadditivity and because C is a partition of T_1, D is a partition of $T_2 \cap T_3$ and V_k is a partition of $T_2 \cup T_3$. The last inequality is satisfied because $\{T_1, T_2 \cup T_3\}$ forms a 2-covering family with $T_1 \cap (T_2 \cup T_3) = T_1 \cap T_2$, and because of our assumption that all 2-covering family inequalities are satisfied. Concluding, we have for all $k \in \{1, \dots, a_2\}$ that

$$w^k b > -v(T_2). \tag{3.12}$$

Finally let $k \in \{1, \dots, a_3\}$. According to Lemma 3.6.5, $\mathcal{Y}(z^k)$ is a partition of T_2. Now first suppose that $T^* \notin \mathcal{Y}(z^k)$. Then

$$z^k b = \sum_{S \in \mathcal{Y}(z^k)} -v(S) > -v(T_2).$$

Here the inequality is satisfied because $\mathcal{Y}(z^k)$ is a partition of T_2, and because T_2 is essential.

Now suppose that $T^* \in \mathcal{Y}(z^k)$. Since $\mathcal{Y}(z^k)$ is a partition of T_2, it follows that $\mathcal{Y}(z^k) \backslash \{T^*\}$ can be split into a partition A of $T_1 \cap T_2$ and a partition

B of $T_2 \cap T_3$. Therefore

$$\begin{aligned} z^k b &= \sum_{S \in \mathcal{Y}(z^k)\setminus\{T^*\}} [-v(S)] - v(N) + v(T_1) + v(T_3) \\ &= \sum_{S\in A}[-v(S)] + \sum_{S\in B}[-v(S)] - v(N) + v(T_1) + v(T_3) \\ &\geq -v(T_1 \cap T_2) - v(T_2 \cap T_3) - v(N) + v(T_1) + v(T_3) \\ &> -v(T_2). \end{aligned}$$

The first inequality follows by superadditivity and the second by our assumption that $v(T_1) + v(T_2) + v(T_3) > v(N) + v(T_1 \cap T_2) + v(T_2 \cap T_3)$.

Concluding, we have for all $k \in \{1, \dots, a_3\}$ that

$$z^k b > -v(T_2). \tag{3.13}$$

Summarising we find for yb that

$$\begin{aligned} yb &= \sum_{k=1}^{a_1} \lambda_k u^k b + \sum_{k=1}^{a_2} \mu_k w^k b + \sum_{k=1}^{a_3} \nu_k z^k b \\ &\geq \sum_{k=1}^{a_2} \mu_k w^k b + \sum_{k=1}^{a_3} \nu_k z^k b \\ &> \sum_{k=1}^{a_2} \mu_k(-v(T_2)) + \sum_{k=1}^{a_3} \nu_k(-v(T_2)) \\ &= -v(T_2). \end{aligned}$$

The first inequality holds because of (3.11). The second inequality is satisfied because of (3.12) and (3.13). The last equality is satisfied since $\sum_{k=1}^{a_2} \mu_k + \sum_{k=1}^{a_3} \nu_k = 1$. □

The next example illustrates the decomposition lemmas that are used in the proof of Theorem 3.5.3. In particular we decompose a specific $y \geq 0$ with $yA = -e(T_2)$ and show that $yb > -v(T_2)$.

Example 3.5.2 Let $N = \{1, \dots, 7\}$ and consider the 3-covering family $\{T_1, T_2, T_3\}$ with $T_1 = \{1, 2\}$, $T_2 = \{2, 3, 4, 5, 6\}$ and $T_3 = \{6, 7\}$. Let $v \in TU^N$ be a chain-component additive game for which all 2-covering family inequalities are satisfied, and for which $v(T_1) + v(T_2) + v(T_3) > v(N) + v(T_1 \cap T_2) + v(T_2 \cap T_3)$ while T_2 is essential.

Now $T^* = N \backslash (T_1 \cup T_3) = \{3, 4, 5\}$ and $\mathcal{T} = \{S \in \mathcal{S} : \{3, 4, 5\} \not\subseteq S\}$. Define the matrix A and the vector b as described in the proof of Theorem 3.5.3. Now let y be given by

$$y_S = \begin{cases} \frac{1}{2}, & \text{if } S = \{1,2,3\}, \{2\}, \{2,3,4\}, \{3,4\}, \{4,5,6\}, \{5,6\}; \\ 1, & \text{if } S = \{1,2,3,4\}; \\ 1\frac{1}{2}, & \text{if } S = \{5,6,7\}, N; \\ 0, & \text{otherwise.} \end{cases}$$

Observe that $yA = \sum_{S \in \mathcal{T} \cup \{T^*\}} y_S(-e(S)) + y_N e(N) = -e(T_2)$. So we need to show that $yb > -v(T_2)$. We will do this by decomposing y. Note that $\mathcal{Y}(y) \backslash \{N\}$ contains a partition of N, for instance $U = \{\{1, 2, 3, 4\}, \{5, 6, 7\}\}$. Therefore we write $y = u^1 + r^1$, with

$$u^1_S = \begin{cases} 1, & \text{if } S = \{1,2,3,4\}, \{5,6,7\}, N; \\ 0, & \text{otherwise.} \end{cases}$$

and

$$r^1_S = \begin{cases} \frac{1}{2}, & \text{if } S = \{1,2,3\}, \{2\}, \{2,3,4\}, \{3,4\}, \{4,5,6\}, \\ & \qquad \{5,6\}, \{5,6,7\}, N; \\ 0, & \text{otherwise.} \end{cases}$$

Now $\mathcal{Y}(r^1) \backslash \{N\}$ does not contain a partition of N. However, it contains a subset that covers each player of $N \backslash T_2$ exactly once, and each player of T_2 exactly twice. For instance $\{\{1, 2, 3\}, \{4, 5, 6\}, \{2\}, \{3, 4\}, \{5, 6, 7\}\}$ is such a subset. Therefore we decompose r^1 into $\frac{1}{2}w^1 + r^2$, with

$$w^1_S = \begin{cases} 1, & \text{if } S = \{1,2,3\}, \{4,5,6\}, \{2\}, \{3,4\}, \{5,6,7\}, N; \\ 0, & \text{otherwise.} \end{cases}$$

and

$$r^2_S = \begin{cases} \frac{1}{2}, & \text{if } S = \{2,3,4\}, \{5,6\}; \\ 0, & \text{otherwise.} \end{cases}$$

Finally we note that $\mathcal{Y}(r^2) = \{\{2, 3, 4\}, \{5, 6\}\}$ is a partition of T_2. Hence, we write $r^2 = \frac{1}{2}z^1$ with

$$z^1_S = \begin{cases} 1, & \text{if } S = \{2,3,4\}, \{5,6\}; \\ 0, & \text{otherwise.} \end{cases}$$

So we have decomposed y into $u^1 + \frac{1}{2}w^1 + \frac{1}{2}z^1$. We will now show that $yb = (u^1 + \frac{1}{2}w^1 + \frac{1}{2}z^1)b > -v(T_2)$. First note that superadditivity of (N, v) implies

$$u^1 b \quad = \quad -v(\{1,2,3,4\}) - v(\{5,6,7\}) + v(N) \geq 0.$$

Furthermore,

$$\begin{aligned} w^1 b &= -v(\{1,2,3\}) - v(\{4,5,6\}) - v(\{2\}) - v(\{3,4\}) \\ &\quad -v(\{5,6,7\}) + v(N) \\ &\geq -v(\{1,2,3,4,5,6\}) - v(\{2,3,4\}) - v(\{5,6,7\}) + v(N) \\ &\geq -v(\{2,3,4\}) - v(\{5,6\}) \\ &> -v(T_2). \end{aligned}$$

Here the first inequality is satisfied due to superadditivity. The second holds because the 2-covering family inequality corresponding to $\{\{1,2,3,4,5,6\}$, $\{5,6,7\}\}$ is satisfied by assumption. The strict inequality is satisfied by our assumption that T_2 is essential. Finally observe that this assumption also proves that

$$z^1 b = -v(\{2,3,4\}) - v(\{5,6\}) > -v(T_2).$$

We conclude that $yb = (u^1 + \frac{1}{2}w^1 + \frac{1}{2}z^1)b > 0 - \frac{1}{2}v(T_2) - \frac{1}{2}v(T_2) = -v(T_2)$. $\diamond$

Let $G = (N, E)$ be a chain. If $T \in \mathcal{S}$ is inessential, then there exists a partition $\{A, B\}$ of T, $A, B \in \mathcal{S}$, with $v(A) + v(B) = v(T)$. Hence, checking whether $T \in \mathcal{S}$ is essential requires the checking of $|T| - 1$ linear equations. So our characterisation of core stability requires the checking of polynomially many linear inequalities and equations. Indeed, in an n-player chain-component additive game there are $\binom{n-1}{2}$ 2-covering and $\binom{n-1}{4}$ 3-covering inequalities, and for each 3-covering inequality that is violated there are at most $n - 3$ linear equations to consider.

The last theorem of this chapter also characterises essential extendibility and core stability. In fact, this theorem reduces the number of linear equations one needs to check in case a 3-covering family inequality is violated.

Theorem 3.5.4 Let $G = (N, E)$ be a chain, and let $v \in TU^N$ be a chain-component additive game with respect to G. The following statements are equivalent:

1. Each 2-covering family inequality is satisfied. For each 3-covering family $\{T_1, T_2, T_3\}$, if $v(A) + v(B) < v(T_2)$ for each partition $\{A, B\}$ of T_2 with $T_1 \cap T_2 \subsetneq A$, $T_2 \cap T_3 \subsetneq B$, and $A, B \in \mathcal{S}$, then the corresponding inequality is satisfied;

2. (N, v) is essential extendible;

3. $C(v)$ is stable.

Proof: Because of Theorem 3.5.3, we only show $3 \Rightarrow 1$. Assume that the core is stable. From Theorem 3.5.3 it follows that each 2-covering family inequality is satisfied. Assume that there is some 3-covering family $\{T_1, T_2, T_3\}$ with $v(T_1) + v(T_2) + v(T_3) > v(N) + v(T_1 \cap T_2) + v(T_2 \cap T_3)$ and $v(A) + v(B) < v(T_2)$ for every partition $\{A, B\}$ of T_2 with $T_1 \cap T_2 \subsetneq A$, $T_2 \cap T_3 \subsetneq B$ and $A, B \in \mathcal{S}$. We show that this leads to a contradiction.

Assume that $\{T_1, T_2, T_3\}$ is a smallest 3-covering family with $v(T_1) + v(T_2) + v(T_3) > v(N) + v(T_1 \cap T_2) + v(T_2 \cap T_3)$ and $v(A) + v(B) < v(T_2)$ for every partition $\{A, B\}$ of T_2 with $T_1 \cap T_2 \subsetneq A$, $T_2 \cap T_3 \subsetneq B$ and $A, B \in \mathcal{S}$ in the following sense: for each 3-covering family $\{S_1, S_2, S_3\}$ with $S_2 \subseteq T_2$ either $v(S_1) + v(S_2) + v(S_3) \leq v(N) + v(S_1 \cap S_2) + v(S_2 \cap S_3)$ or $v(A) + v(B) = v(S_2)$ for some partition $\{A, B\}$ of S_2 with $S_1 \cap S_2 \subsetneq A$, $S_2 \cap S_3 \subsetneq B$ and $A, B \in \mathcal{S}$.

Since the core is stable, it follows from Theorem 3.5.3 that T_2 is inessential. Hence, there is a partition $\{A, B\}$ of T_2, $A, B \in \mathcal{S}$ with $v(A) + v(B) = v(T_2)$. By assumption, either $A \subseteq T_1 \cap T_2$ or $B \subseteq T_2 \cap T_3$. Without loss of generality assume that $A \subseteq T_1 \cap T_2$.

First suppose that $A = T_1 \cap T_2$. Then obviously $B = T_2 \backslash T_1$. Consequently

$$\begin{aligned} v(T_1) + v(T_2) + v(T_3) &= v(T_1) + v(T_1 \cap T_2) + v(T_2 \backslash T_1) + v(T_3) \\ &\leq v(T_1 \cup T_2) + v(T_1 \cap T_2) + v(T_3) \\ &\leq v(N) + v(T_1 \cap T_2) + v(T_2 \cap T_3). \end{aligned}$$

The first inequality holds because of superadditivity and the second because $\{T_1 \cup T_2, T_3\}$ is a 2-covering family with $(T_1 \cup T_2) \cap T_3 = T_2 \cap T_3$. Since $v(T_1) + v(T_2) + v(T_3) \leq v(N) + v(T_1 \cap T_2) + v(T_2 \cap T_3)$ is a contradiction to our assumption, we conclude that $A \neq T_1 \cap T_2$.

Now suppose that $A \subsetneq T_1 \cap T_2$. Observe that

$$\begin{aligned} v(T_1) + v(B) + v(T_3) &= v(T_1) + v(A) + v(B) + v(T_3) - v(A) \\ &= v(T_1) + v(T_2) + v(T_3) - v(A) \\ &> v(N) + v(T_1 \cap T_2) + v(T_2 \cap T_3) - v(A) \\ &\geq v(N) + v((T_1 \cap T_2) \backslash A) + v(T_2 \cap T_3) \\ &= v(N) + v(T_1 \cap B) + v(B \cap T_3). \end{aligned}$$

The first inequality holds by assumption and the second one because of superadditivity. The last equality comes from $(T_1 \cap T_2) \backslash A = T_1 \cap B$ and $T_2 \cap T_3 = B \cap T_3$. Obviously, $\{T_1, B, T_3\}$ is a 3-covering family with $B \subseteq T_2$ and $v(T_1) + v(B) + v(T_3) > v(N) + v(T_1 \cap B) + v(B \cap T_3)$. By assumption, there is some partition $\{C, D\}$ of B with $T_1 \cap B \subsetneq C$, $B \cap T_3 \subsetneq D$, $C, D \in \mathcal{S}$ and $v(C) + v(D) = v(B)$. From $v(A) + v(B) = v(T_2)$ and $v(C) + v(D) = v(B)$ it follows that $v(A) + v(C) + v(D) = v(T_2)$. From superadditivity we conclude that $v(A \cup C) + v(D) = v(T_2)$. Note that $B \cap T_3 \subsetneq D$ and therefore $T_2 \cap T_3 \subsetneq D$. Furthermore, since $T_1 \cap B \subsetneq C$, we have that $T_1 \cap T_2 \subsetneq A \cup C$. This contradicts our initial assumption. $\square$

3.6 Proofs of lemmas

In this section we prove the decomposition lemmas needed for the proof of Theorem 3.5.3. Furthermore, we prove some auxiliary lemmas. Throughout this section we use the notation introduced in the proof of Theorem 3.5.3.

Lemma 3.6.1 Let $y \geq 0$ be such that $yA = -e(T_2)$. Then $y = \sum_{k=1}^{a_1} \lambda_k u^k + r^1$, with $r^1 \geq 0$, $r^1 A = -e(T_2)$, $\mathcal{Y}(r^1) \backslash \{N\}$ does not contain a partition of N, and for all $k \in \{1, \ldots, a_1\}$, $\lambda_k > 0$ and u^k satisfying

(A1) $u^k_S \in \{0, 1\}$ for all $S \in \mathcal{T} \cup \{T^*, N\}$, $u^k A = 0$ and $\mathcal{Y}(u^k) = U_k \cup \{N\}$ for some partition U_k of N.

Proof: Let $y \geq 0$ be such that $yA = -e(T_2)$. We show the decomposition by recursion. Suppose that for some $a^* \geq 0$ we have written $y = \sum_{k=1}^{a^*} \lambda_k u^k + r^1$, with $r^1 \geq 0$, $r^1 A = -e(T_2)$ and for all $k \in \{1, \ldots, a^*\}$, $\lambda_k > 0$ and u^k satisfies (A1). Note that this certainly holds for $a^* = 0$ and $r^1 = y$.

Now if $\mathcal{Y}(r^1) \backslash \{N\}$ does not contain a partition of N, then we are done, so suppose that $\mathcal{Y}(r^1) \backslash \{N\}$ contains a partition, say U, of N. Define

- $u_S^{a^*+1} = 1$ if $S \in U \cup \{N\}$;
- $u_S^{a^*+1} = 0$ if $S \notin U \cup \{N\}$;
- $\lambda_{a^*+1} = \min\{r_S^1 : S \in U \cup \{N\}\}$.

Note that $\lambda_{a^*+1} > 0$ and that $\mathcal{Y}(u^{a^*+1}) = U \cup \{N\}$. Observe, since U is a partition of N, that $u^{a^*+1} A = 0$. Thus, u^{a^*+1} satisfies (A1). Furthermore, by definition of λ_{a^*+1} and u^{a^*+1}, $\bar{r}^1 = r^1 - \lambda_{a^*+1} u^{a^*+1} \geq 0$. Finally, note, because $u^k A = 0$ for all $k \in \{1, \ldots, a^* + 1\}$ and because $yA = -e(T_2)$, that $\bar{r}^1 A = yA - \sum_{k=1}^{a^*+1} \lambda_k u^k A = -e(T_2)$.

So $y = \sum_{k=1}^{a^*+1} \lambda_k u^k + \bar{r}^1$, with $\bar{r}^1 \geq 0$, $\bar{r}^1 A = -e(T_2)$, and for all $k \in \{1, \ldots, a^* + 1\}$, $\lambda_k > 0$ and u^k satisfies (A1).

Observe that because of our choice of λ_{a^*+1}, $\mathcal{Y}(\bar{r}^1) \subsetneq \mathcal{Y}(r^1)$. This implies that in a finite number of steps we can decompose y into $\sum_{k=1}^{a_1} \lambda_k u^k + r^1$, with $r^1 \geq 0$, $r^1 A = -e(T_2)$, $\mathcal{Y}(r^1) \backslash \{N\}$ does not contain a partition of N, and for all $k \in \{1, \ldots, a_1\}$, $\lambda_k > 0$ and u^k satisfies (A1). $\square$

Lemma 3.6.2 Let $r^2 \geq 0$ be such that $r^2 A = f(-e(T_2))$ for some $f \in \mathbb{R}$ with $0 < f \leq 1$. Then, $\sum_{S \in \mathcal{Y}(r^2) \backslash \{N\} : a \in S} r_S^2 \geq \sum_{S \in \mathcal{Y}(r^2) \backslash \{N\} : a-1 \in S} r_S^2$ for all $a \in T_1 \cup T_2$ with $a > 1$. Furthermore, $\sum_{S \in \mathcal{Y}(r^2) \backslash \{N\} : a+1 \in S} r_S^2 = \sum_{S \in \mathcal{Y}(r^2) \backslash \{N\} : a \in S} r_S^2$ for all $a \in T_3 \backslash T_2$ with $a < n$.

Proof: Let $r^2 \geq 0$ be such that $r^2 A = f(-e(T_2))$ for some $f \in \mathbb{R}$ with $0 < f \leq 1$. Let $a \in T_1 \cup T_2$ with $a > 1$.

If $a \in T_1 \backslash T_2$, then it follows that $a - 1 \in T_1 \backslash T_2$. It follows from $r^2 A = f(-e(T_2))$ that

$$\sum_{S \in \mathcal{Y}(r^2) \backslash \{N\} : a \in S} r_S^2 = r_N^2 = \sum_{S \in \mathcal{Y}(r^2) \backslash \{N\} : a-1 \in S} r_S^2.$$

If $a \in T_2$ and $a-1 \in T_1 \backslash T_2$, then it follows from $r^2 A = f(-e(T_2))$ that

$$\sum_{S \in \mathcal{Y}(r^2) \backslash \{N\} : a \in S} r_S^2 = f + r_N^2 > r_N^2 = \sum_{S \in \mathcal{Y}(r^2) \backslash \{N\} : a-1 \in S} r_S^2.$$

Finally, if $a-1 \in T_2$, then it follows from $r^2 A = f(-e(T_2))$ that

$$\sum_{S \in \mathcal{Y}(r^2) \backslash \{N\} : a \in S} r_S^2 = f + r_N^2 = \sum_{S \in \mathcal{Y}(r^2) \backslash \{N\} : a-1 \in S} r_S^2.$$

So we conclude that

$$\sum_{S \in \mathcal{Y}(r^2) \backslash \{N\} : a \in S} r_S^2 \geq \sum_{S \in \mathcal{Y}(r^2) \backslash \{N\} : a-1 \in S} r_S^2.$$

Similarly it can be shown for all $a \in T_3 \backslash T_2$ with $a < n$ that

$$\sum_{S \in \mathcal{Y}(r^2) \backslash \{N\} : a+1 \in S} r_S^2 = \sum_{S \in \mathcal{Y}(r^2) \backslash \{N\} : a \in S} r_S^2. \qquad \square$$

Lemma 3.6.3 Let $r^2 \geq 0$ be such that $r^2 A = f(-e(T_2))$ for some $f \in \mathbb{R}$ with $0 < f \leq 1$, $r_2^N > 0$ and $\mathcal{Y}(r^2) \backslash \{N\}$ does not contain a partition of N. Then $\mathcal{Y}(r^2)$ contains a partition U of $T_1 \cup T_2$ and a partition V of $T_2 \cup T_3$ with $U \cap V = \emptyset$.

Proof: We will show how to obtain a partition of $T_1 \cup T_2$. Analogously one can find a partition of $T_2 \cup T_3$. First we will show that we can find disjoint elements $S_k \in \mathcal{Y}(r^2)$, $k \in \{1, \ldots, q\}$, such that $T_1 \cup T_2 \subseteq \bigcup_{k=1}^q S_k$. We will do this by giving a recursive argument.

Because $r^2 A = f(-e(T_2))$ for some $f \in \mathbb{R}$ with $0 < f \leq 1$ and $1 \notin T_2$, we have $\sum_{S \in \mathcal{T} \cup \{T^*\} : 1 \in S} r_S^2 = r_N^2$. By assumption $r_N^2 > 0$ and we conclude that $\sum_{S \in \mathcal{T} \cup \{T^*\} : 1 \in S} r_S^2 > 0$. Hence, there exists an $S_1 \in \mathcal{Y}(r^2)$, with $1 \in S_1$.

Now suppose that we have selected disjoint $S_k \in \mathcal{Y}(r^2)$, $k \in \{1, \ldots, t\}$, such that $N \backslash (\bigcup_{k=1}^t S_k) = \{b, \ldots, n\}$ for some $b \in N$. Note that $t = 1$ and S_1 satisfy this property.

If $b \notin T_1 \cup T_2$, then we are done, so suppose that $b \in T_1 \cup T_2$. According to Lemma 3.6.2, with $a = b$, $\sum_{S \in \mathcal{Y}(r^2) \backslash \{N\} : b \in S} r_S^2 \geq \sum_{S \in \mathcal{Y}(r^2) \backslash \{N\} : b-1 \in S} r_S^2$. Since $b-1 \in S_t$, $b \notin S_t$ and $S_t \in \mathcal{Y}(r^2)$, there is an $S_{t+1} \in \mathcal{Y}(r^2)$ with

$b-1 \notin S_{t+1}$ and $b \in S_{t+1}$. We conclude that $N \setminus \bigcup_{k=1}^{t+1} S_k = \{c, \ldots, n\}$ with $c > b$. By recursion we obtain disjoint $S_k \in \mathcal{Y}(r^2)$, $k \in \{1, \ldots, q\}$, with $T_1 \cup T_2 \subseteq \bigcup_{k=1}^{q} S_k$.

We will now show that $T_1 \cup T_2 = \bigcup_{k=1}^{q} S_k$ by contradiction. Suppose that $T_1 \cup T_2 \subsetneq \bigcup_{k=1}^{q} S_k$. Then $N \backslash (\bigcup_{k=1}^{q} S_k) = \{b, \ldots, n\}$ for some $b \in T_3 \backslash T_2$ with $b-1 \in T_3 \backslash T_2$. According to Lemma 3.6.2, with $a = b-1$ it follows that $\sum_{S \in \mathcal{Y}(r^2) \backslash \{N\}: b \in S} r_S^2 = \sum_{S \in \mathcal{Y}(r^2) \backslash \{N\}: b-1 \in S} r_S^2$. Since $b-1 \in S_q$, $b \notin S_q$ and $S_q \in \mathcal{Y}(r^2)$, it follows that there is an $S_{q+1} \in \mathcal{Y}(r^2) \backslash \{N\}$ with $b-1 \notin S_{q+1}$ and $b \in S_{q+1}$. Note that $N \backslash (\bigcup_{k=1}^{q+1} S_k) = \{c, \ldots, n\}$ with $c > b$. By recursion we therefore obtain a partition of N. However, initially we assumed that $\mathcal{Y}(r^2) \backslash \{N\}$ does not contain a partition of N. From this contradiction we conclude that $T_1 \cup T_2 = \bigcup_{k=1}^{q} S_k$.

Now let $U = \{S_1, \ldots, S_q\} \subseteq \mathcal{Y}(r^2)$ be a partition of $T_1 \cup T_2$ such that for all $a \in S_i$ and $b \in S_j$ it is satisfied that $a < b$ if $i < j$. Similarly, let $V = \{R_1, \ldots, R_m\} \subseteq \mathcal{Y}(r^2)$ be a partition of $T_2 \cup T_3$ such that for all $a \in R_i$ and $b \in R_j$ it is satisfied that $a < b$ if $i < j$. If $U \cap V \neq \emptyset$, then $S_i = R_j$ for some $i \in \{1, \ldots, q\}$, $j \in \{1, \ldots, m\}$. Hence, $\{S_1, \ldots, S_i, R_{j+1}, \ldots, R_m\}$ is a partition of N. This contradicts the assumption that $\mathcal{Y}(r^2) \setminus \{N\}$ does not contain a partition of N. We conclude that $U \cap V = \emptyset$. □

Lemma 3.6.4 Let $y \geq 0$ be such that $yA = -e(T_2)$ and $\mathcal{Y}(y) \backslash \{N\}$ does not contain a partition of N. Then $y = \sum_{k=1}^{a_2} \mu_k w^k + r^2$, with $r^2 \geq 0$, $r_N^2 = 0$, $\sum_{k=1}^{a_2} \mu_k \leq 1$ and for all $k \in \{1, \ldots, a_2\}$, $\mu_k > 0$ and w^k satisfies

(A2) $w_S^k \in \{0, 1\}$ for all $S \in \mathcal{T} \cup \{T^*, N\}$, $w^k A = -e(T_2)$ and $\mathcal{Y}(w^k) = U_k \cup V_k \cup \{N\}$ for some partition U_k of $T_1 \cup T_2$ and some partition V_k of $T_2 \cup T_3$ with $U_k \cap V_k = \emptyset$.

Proof: Let $y \geq 0$ be such that $yA = -e(T_2)$, and such that $\mathcal{Y}(y) \backslash \{N\}$ does not contain a partition of N. We show the decomposition recursively. Suppose that for some $a^* \geq 0$ we have written $y = \sum_{k=1}^{a^*} \mu_k w^k + r^2$, with $r^2 \geq 0$, $\sum_{k=1}^{a^*} \mu_k \leq 1$ and for all $k \in \{1, \ldots, a^*\}$, $\mu_k > 0$ and w^k satisfies (A2). Note that this certainly holds for $a^* = 0$ and $r^2 = y$. If $r_N^2 = 0$ then we are done, so suppose that $r_N^2 > 0$. Observe that since $\mathcal{Y}(y) \backslash \{N\}$ does not contain a partition of N, and since $\mathcal{Y}(r^2) \subseteq \mathcal{Y}(y)$, it follows that $\mathcal{Y}(r^2)\backslash\{N\}$ does not contain a partition of N.

We will first show that $\sum_{k=1}^{a^*} \mu_k < 1$ by contradiction. Suppose that $\sum_{k=1}^{a^*} \mu_k = 1$. Then it follows, using $yA = -e(T_2)$ and $w^k A = -e(T_2)$ for all $k \in \{1, \ldots, a^*\}$, that $r^2 A = yA - \sum_{k=1}^{a^*} \mu_k w^k A = (1 - \sum_{k=1}^{a^*} \mu_k)(-e(T_2)) = 0$. Since $r_N^2 > 0$, it follows that $\mathcal{Y}(r^2)\backslash\{N\}$ is a balanced collection of N. From Lemma 3.2.1 we conclude that $\mathcal{Y}(r^2)\backslash\{N\}$ contains a partition of N, which contradicts the fact that $\mathcal{Y}(r^2)\backslash\{N\}$ does not contain a partition of N. We conclude that $\sum_{k=1}^{a^*} \mu_k < 1$.

Since $r^2 A = (1 - \sum_{k=1}^{a^*} \mu_k)(-e(T_2))$ with $\sum_{k=1}^{a^*} \mu_k < 1$, $r_N^2 > 0$ and $\mathcal{Y}(r^2)\backslash\{N\}$ does not contain a partition of N, it follows that r^2 satisfies the conditions of Lemma 3.6.3. According to Lemma 3.6.3, $\mathcal{Y}(r^2)$ contains a partition U_{a^*+1} of $T_1 \cup T_2$ and a partition V_{a^*+1} of $T_2 \cup T_3$ with $U_{a^*+1} \cap V_{a^*+1} = \emptyset$.

Define

- $w_S^{a^*+1} = 1$ if $S \in U_{a^*+1} \cup V_{a^*+1} \cup \{N\}$;
- $w_S^{a^*+1} = 0$ if $S \notin U_{a^*+1} \cup V_{a^*+1} \cup \{N\}$;
- $\mu_{a^*+1} = \min\{r_S^2 : S \in U_{a^*+1} \cup V_{a^*+1} \cup \{N\}\}$.

Note that $\mu_{a^*+1} > 0$. We will now show that $w^{a^*+1} A = -e(T_2)$. For each $i \in T_2$ there are unique $S \in U_{a^*+1}$ and $T \in V_{a^*+1}$ with $i \in S$ and $i \in T$. Note that since $U_{a^*+1} \cap V_{a^*+1} = \emptyset$, $S \neq T$. Furthermore, for each $i \in T_1 \backslash T_2$ there is a unique $S \in U_{a^*+1}$ with $i \in S$ and for each $i \in T_3 \backslash T_2$ there is a unique $T \in V_{a^*+1}$ with $i \in T$. We conclude that $w^{a^*+1} A = -e(T_2)$.

Since $w^{a^*+1} A = -e(T_2)$ we now observe that w^{a^*+1} satisfies (A2). Also note that $\bar{r}^2 = r^2 - \mu_{a^*+1} w^{a^*+1} \geq 0$. We will now show by contradiction that $\sum_{k=1}^{a^*+1} \mu_k \leq 1$. Suppose that $\sum_{k=1}^{a^*+1} \mu_k > 1$. Because $\sum_{k=1}^{a^*} \mu_k \leq 1$, it follows that there is a $d \in \mathbb{R}$ with $0 \leq d < \mu_{a^*+1}$ with $\sum_{k=1}^{a^*} \mu_k + d = 1$. Trivially, $y = \sum_{k=1}^{a^*} \mu_k w^k + d w^{a^*+1} + (r^2 - d w^{a^*+1})$. By definition of d, $(r^2 - d w^{a^*+1}) \gneq (r^2 - \mu_{a^*+1} w^{a^*+1}) \geq 0$. Since $yA = -e(T_2)$, $w^k A = -e(T_2)$ for all $k \in \{1, \ldots, a^* + 1\}$ and $\sum_{k=1}^{a^*} \mu_k + d = 1$, it follows that $(r^2 - d w^{a^*+1}) A = 0$. Because $d < \mu_{a^*+1}$, $r_N^2 - d w_N^{a^*+1} > 0$. Therefore it follows that $\mathcal{Y}(r^2 - dw^{a^*+1}) \neq \emptyset$ is a balanced collection. By Lemma 3.2.1 it now follows that $\mathcal{Y}(r^2 - dw^{a^*+1})\backslash\{N\}$ contains a partition of N. Since $\mathcal{Y}(r^2 - dw^{a^*+1}) \subseteq \mathcal{Y}(r^2)$, $\mathcal{Y}(r^2)\backslash\{N\}$ contains a partition of N. This is clearly a contradiction to our initial assumption, so we conclude that $\sum_{k=1}^{a^*+1} \mu_k \leq 1$.

It follows that $y = \sum_{k=1}^{a^*+1} \mu_k w^k + \bar{r}^2$, with $\bar{r}^2 \geq 0$, $\sum_{k=1}^{a^*+1} \mu_k \leq 1$ and for all $k \in \{1, \ldots, a^* + 1\}$, $\mu_k > 0$ and w^k satisfies (A2).

Observe that because of our choice of μ_{a^*+1}, $\mathcal{Y}(\bar{r}^2) \subsetneq \mathcal{Y}(r^2)$. This implies that in a finite number of steps we can decompose y into $\sum_{k=1}^{a_2} \mu_k w^k + r^2$, with $r^2 \geq 0$, $r_N^2 = 0$, $\sum_{k=1}^{a_2} \mu_k \leq 1$ and for all $k \in \{1, \ldots, a_2\}$, $\mu_k > 0$ and w^k satisfies (A2). □

Lemma 3.6.5 Let $y \geq 0$ be such that $y_N = 0$ and $yA = d(-e(T_2))$, for some $d \in \mathbb{R}$ with $0 \leq d \leq 1$. Then $y = \sum_{k=1}^{a_3} \nu_k z^k$ with $\sum_{k=1}^{a_3} \nu_k = d$, and for all $k \in \{1, \ldots, a_3\}$, $\nu_k > 0$ and z^k satisfies

(A3) $z_S^k \in \{0, 1\}$ for all $S \in \mathcal{T} \cup \{T^*, N\}$, $z^k A = -e(T_2)$ and $\mathcal{Y}(z^k) = U_k$ for some partition U_k of T_2.

Proof: Let $y \geq 0$ be such that $y_N = 0$, and $yA = d(-e(T_2))$ for some $d \in \mathbb{R}$ with $0 \leq d \leq 1$. We recursively show the decomposition. Suppose that for some $a^* \geq 0$ we have written $y = \sum_{k=1}^{a^*} \nu_k z^k + r^3$, with $\sum_{k=1}^{a^*} \nu_k \leq d$, $r^3 \geq 0$ and for all $k \in \{1, \ldots, a^*\}$, $\nu_k > 0$ and that z^k satisfies (A3). Note that this certainly holds for $a^* = 0$ and $r^3 = y$.

Now if $\sum_{k=1}^{a^*} \nu_k = d$, then it follows, because $yA = d(-e(T_2))$ and $z^k A = -e(T_2)$ for all $k \in \{1, \ldots, a^*\}$, that $r^3 A = yA - \sum_{k=1}^{a^*} \nu_k z^k A = 0$. Because $r_N^3 = 0$, $r^3 \geq 0$ and because A has only non-positive entries in each row that does not correspond to N with at least one negative entry, we conclude that $r^3 = 0$. So $y = \sum_{k=1}^{a^*} \nu_k z^k$ and we are done. Therefore suppose that $\sum_{k=1}^{a^*} \nu_k < d$.

Now $r^3 A = yA - \sum_{k=1}^{a^*} \nu_k z^k A = (d - \sum_{k=1}^{a^*} \nu_k)(-e(T_2))$, with $d - \sum_{k=1}^{a^*} \nu_k > 0$. Since $r_N^3 = 0$, and because in A the only row with positive entries is the row corresponding to N, this means that $r_S^3 = 0$ for all $S \in \mathcal{T} \cup \{T^*, N\}$ with $S \not\subseteq T_2$. This implies that $\mathcal{Y}(r^3)$ is a balanced collection on T_2. From Lemma 3.2.1 it follows that $\mathcal{Y}(r^3)$ contains a partition of T_2. Now let U be such a partition. Define

- $z_S^{a^*+1} = 1$ if $S \in U$;
- $z_S^{a^*+1} = 0$ if $S \notin U$;
- $\nu_{a^*+1} = \min\{r_S^3 : S \in U\}$.

Note that $\nu_{a^*+1} > 0$. Since $z^{a^*+1}A = -e(T_2)$, it follows that z^{a^*+1} satisfies (A3). Also observe that by definition of ν_{a^*+1} and z^{a^*+1}, $\bar{r}^3 = r^3 - \nu_{a^*+1}z^{a^*+1} \geq 0$. It remains to show that $\sum_{k=1}^{a^*+1} \nu_k \leq d$.

Suppose that $\sum_{k=1}^{a^*+1} \nu_k > d$. Then it follows that $\bar{r}^3 A = (d - \sum_{k=1}^{a^*+1} \nu_k)(-e(T_2))$, where $d - \sum_{k=1}^{a^*+1} \nu_k < 0$. Hence, $\bar{r}^3 A = fe(T_2)$ for some $f > 0$. However, this is impossible, since $\bar{r}^3 \geq 0$, $\bar{r}_N^3 = 0$ and because A contains only non-positive entries in the rows not corresponding to N. Therefore we obtain that $\sum_{k=1}^{a^*+1} \nu_k \leq d$.

Hence, we have that $y = \sum_{k=1}^{a^*+1} \nu_k z^k + \bar{r}^3$, with $\sum_{k=1}^{a^*+1} \nu_k \leq d$, $\bar{r}^3 \geq 0$ and for all $k \in \{1, \ldots, a^*+1\}$, $\nu_k > 0$ and z^k satisfies (A3).

Observe that by definition of ν_{a^*+1} and z^{a^*+1}, $\mathcal{Y}(\bar{r}^3) \subsetneq \mathcal{Y}(r^3)$. Hence, in a finite number of steps we obtain that $y = \sum_{k=1}^{a_3} \nu_k z^k$, where $\nu_k > 0$ and z^k satisfies (A3) for all $k \in \{1, \ldots, a_3\}$. Since $yA = d(-e(T_2))$ and $z^k A = -e(T_2)$ for all $k \in \{1, \ldots, a_3\}$ it follows that $\sum_{k=1}^{a_3} \nu_k = d$. □

Lemma 3.6.6 Let $k \in \{1, \ldots, a_2\}$ and let $\mathcal{Y}(w_k) = U_k \cup V_k \cup \{N\}$ with U_k a partition of $T_1 \cup T_2$ and V_k a partition of $T_2 \cup T_3$. Let $\bar{V}_k = \{T \in V_k : T \nsubseteq T_2\}$. Then $\bar{V}_k \neq \emptyset$ and $V_k \backslash \bar{V}_k \neq \emptyset$. Furthermore, $T \cap T_1 = \emptyset$ for all $T \in \bar{V}_k$ and there is a $T \in \bar{V}_k$ with $T \cap T_2 \neq \emptyset$.

Proof: Note that $\bar{V}_k \neq \emptyset$, since V_k is a partition of $T_2 \cup T_3$ and $T_3 \backslash T_2 \neq \emptyset$.

We will now show that for all $T \in \bar{V}_k$, $T_1 \cap T = \emptyset$ by contradiction. Suppose that there is a $T \in \bar{V}_k$ with $T_1 \cap T \neq \emptyset$. Since $T \in \bar{V}_k$ it follows that $T \nsubseteq T_2$. That is, there is a $j \in T$ with $j \in T_3 \backslash T_2$. Since T is connected it follows that $T^* \subsetneq T$. This is a contradiction since the coalitions containing T^* are not in $\mathcal{T}$ and therefore also not in $\mathcal{Y}(r^2) \backslash \{N\}$. Hence, for all $T \in \bar{V}_k$, $T \cap T_1 = \emptyset$. Because $T_1 \cap T_2 \neq \emptyset$, there is an $S \in V_k$ with $S \cap T_1 \neq \emptyset$. This implies that $S \notin \bar{V}_k$ and hence that $V_k \neq \bar{V}_k$. Finally, we prove that there is a $T \in \bar{V}_k$ with $T \cap T_2 \neq \emptyset$. Suppose that for all $T \in \bar{V}_k$, $T \cap T_2 = \emptyset$. According to Lemma 3.6.3, $\mathcal{Y}(r^2)$ contains a partition U of $T_1 \cup T_2$. This implies that $U \cup \bar{V}_k$ forms a partition of N, contradicting our initial assumption. $\square$

Chapter 4

Dominating set games

4.1 Introduction

A *domination problem* consists of a given graph $G = (V, E)$, a positive integer $k \in \mathbb{N}$, and a non-negative function $w : V \to \mathbb{R}_+$ that assigns a fixed cost to each vertex. A k-dominating set is a set $D \subseteq V$ such that the distance between any vertex in V and at least one vertex in D is at most k. The k-domination problem is the problem of finding a so-called minimum weighted k-dominating set of G, i.e. a k-dominating set that minimises the total cost of its vertices.

Domination problems are widely studied in graph theory. Meir and Moon (1975) investigate domination problems on trees. Some results of Meir and Moon (1975) are extended to larger classes of graphs in Farber (1981). In Haynes, Hedetniemi, and Slater (1998) an overview of literature on domination problems is given.

An illustration of domination problems is the following example. A number of regions is discussing the placement of several facilities within their regions. The placement of a facility in a region entails a certain fixed cost. Therefore, the regions decide to restrict the number of facilities that will be placed. However, each region demands that a facility is placed within a reasonable distance. The problem of placing the facilities at minimum cost can now be regarded as a domination problem. Let $G = (V, E)$ be the graph where regions correspond to vertices, and where edges represent pairs of neighbouring regions. The proximity condition of the regions can be described by an integer k and the cost of placing the facilities by a map $w : V \to \mathbb{R}_+$. Placing the facilities at minimum cost is now equivalent to finding a minimum weighted k-dominating set on G.

A natural question that now arises is how to allocate the total costs of placing the facilities among the participating regions. This chapter, which is based on Van Velzen (2004a), uses game theory to study this problem. We introduce three cost games that model this cost allocation problem.

Our three dominating set games have in common that the cost of the grand coalition N coincides with the minimum weighted k-domination number. However, coalitions are allowed different possibilities of placing the facilities in each of the three games. In the *relaxed dominating set game* the vertices in the graph are considered to be public vertices. Specifically, we allow coalitions to place facilities in regions corresponding to non-members. Furthermore, coalitions are allowed to use every edge present in the graph in order to meet the proximity condition of its members.

For the *intermediate dominating set game* we assume that the vertices in the graph are private. It might, for instance, be the case that regions outside the coalition can block the placement of a facility within their region. So coalitions are forced to place the facilities in their own regions. However, we still assume that coalitions are allowed to use all the edges present in the graph in order to meet the proximity condition of its members.

The last situation we consider also contains private vertices. However, in this situation, coalitions are only allowed to use those edges with both endpoints being member of the coalition. The resulting game is called the *rigid dominating set game*.

In spite of the differences between these three games, we will obtain a common necessary and sufficient condition for non-emptiness of their cores. In particular, if one of the dominating set games possesses core elements, then the other two dominating set games possess core elements as well. We also derive relations between the cores of the dominating set games and we present a class of graphs for which the corresponding dominating set games have a non-empty core for all cost functions $w : V \to \mathbb{R}_+$ and all $k \in \mathbb{N}$.

Finally we study concavity.

Other game theoretical approaches to location problems include facility location games (Kolen and Tamir (1990), Tamir (1992)) and minimum spanning forest games (Granot and Granot (1992)). In Kolen and Tamir (1990) and Tamir (1992) a general class of facility location problems is studied. They mainly focus on trees, and for these graphs non-emptiness of the core is established. In Granot and Granot (1992) no restrictions are made on the proximity of the facilities. In case the underlying graph is a tree, non-emptiness of the core is shown.

The remainder of this chapter is organised as follows. In Section 4.2 we recall some basic concepts and notation from graph theory. In Section 4.3 we introduce three dominating set games. In Section 4.4 we study the cores of the dominating set games. Finally, in Section 4.5 we study concavity.

4.2 Stars, substars and dominating sets

In this section we introduce some basic concepts and notation from graph theory. We conclude the section with the definition of combinatorial optimisation games.

Let $G = (V, E)$ be a graph. The *distance* $d_G(v, w)$ between $v, w \in V$ is the length of a shortest (v, w)-path. If $v, w \in V$ are not connected via any path, then $d_G(v, w) = \infty$. The *eccentricity* of $v \in V$ is the maximum distance to v in G, i.e. $e_G(v) = \max\{d_G(v, x) : x \in V\}$. The *diameter* $\Delta(G)$ is the maximum over all eccentricities, i.e. $\Delta(G) = \max\{e_G(v) : v \in V\}$. Observe that $\Delta(G)$ is equal to the maximum distance within G. The *radius* $r(G)$ is the minimum over all eccentricities, i.e. $r(G) = \min\{e_G(v) : v \in V\}$. For each vertex $v \in V$, the *k-neighbourhood of* v, denoted by $N_k(v)$, consists of the vertices at distance at most k of v, i.e. $N_k(v) = \{w \in V : d_G(v, w) \leq k\}$. The k-neighbourhood of v is also called the *k-star at v*. For technical purposes, we introduce two other star-like concepts. If $T \subseteq N_k(v)$ contains v, then T is called a *k-substar at v*. The set of k-substars at $j \in V$ is denoted by $\mathcal{S}_k(j)$, i.e. $\mathcal{S}_k(j) = \{S \subseteq V : S \text{ is a } k\text{−substar at } j\}$. If $T \subseteq N_k(v)$, $T \neq \emptyset$, is such that $d_{G_T}(v, x) \leq k$ for all $x \in T$, then T is called a *proper k-*

substar at v. Note that if T is a proper k-substar at v, then G_T is necessarily connected and contains v. The set of proper k-substars at $j \in V$ is denoted by $\mathcal{P}_k(j)$, i.e. $\mathcal{P}_k(j) = \{S \subseteq V : S \text{ is a proper } k\text{−substar at } j\}$.

Example 4.2.1 Let $G = (V, E)$ be the graph depicted in Figure 4.1 and let $k = 2$. Then $\{1, 2, 3\}$ is a 2-star at 1 and $\{1, 3\}$ is a 2-substar at 1. However, $\{1, 3\}$ is not a proper 2-substar at 1, since $G_{\{1,3\}}$ is a disconnected subgraph and therefore $d_{G_{\{1,3\}}}(1, 3) = \infty$. Also note that $\{1, 3\}$ is not a 1-substar at 2 because it does not contain 2 itself. $\diamond$

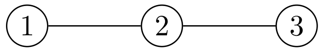

Figure 4.1: The set $\{1, 3\}$ is a 2-substar at 1.

Let $G = (V, E)$ be a graph and let $T \subseteq V$. Let $K \subseteq T$ be such that $\bigcup_{v \in K} N_k^{G_T}(v) = T$, where $N_k^{G_T}(v)$ is the k-neighbourhood of v in G_T. For future purposes we now show that we can partition T into disjoint proper k-substars at v, $v \in K$. We do this by assigning each vertex in $T \backslash K$ to exactly one vertex in K. In particular, we assign $v \in T \backslash K$ to the closest vertex in K. In case of a tie, we pick the vertex in K with lowest index number. So write $K = \{v_1, \ldots, v_m\}$. Define for each $v \in V$, $A(v) = \{l \in \{1, \ldots, m\} : d_{G_T}(v, v_l) \leq d_{G_T}(v, v_i) \text{ for each } i \in \{1, \ldots, m\}\}$ to be the index set of vertices of K that are closest to v. Then, for each $l \in \{1, \ldots, m\}$, let $U_l = \{v \in V : l = \min A(v)\}$.

Lemma 4.2.1 The sets U_l, $l \in \{1, \ldots, m\}$, are proper k-substars at v_l that partition T.

Proof: By definition of U_l, $l \in \{1, \ldots, m\}$, it is satisfied that the sets U_l, $l \in \{1, \ldots, m\}$, form a partition of T. Therefore we only show that each U_l, $l \in \{1, \ldots, m\}$, forms a proper k-substar at v_l. Let $l \in \{1, \ldots, m\}$ and let $q \in U_l$. Let P be the vertex set of a shortest (q, v_l)-path in G_T. We show that U_l is a proper k-substar at v_l by showing that $z \in U_l$ for each $z \in P \backslash \{q, v_l\}$.

Let $z \in P \backslash \{q, v_l\}$ and let $i \in \{1, \ldots, m\}$. Then

$$\begin{aligned} d_{G_T}(z, v_l) &= d_{G_T}(q, v_l) - d_{G_T}(q, z) \\ &\leq d_{G_T}(q, v_i) - d_{G_T}(q, z) \\ &\leq d_{G_T}(z, v_i). \end{aligned}$$

The equality is satisfied because a shortest path from q to v_l passes through z. The first inequality holds because $q \in U_l$. Note that this inequality can only be an equality in case $l \leq i$. The second inequality is satisfied due to the triangle inequality.

We now note that the distance between z and v_l in G_T is at most the distance between z and v_i, and that equality can only occur in case $l \leq i$. We conclude that $z \in U_l$. □

Let $G = (V, E)$ be a graph and let $k \in \mathbb{N}$. A set $D \subseteq V$ is called a *k-dominating set* if each $v \in V$ is at distance at most k from a vertex in D. Formally, D is a k-dominating set if for all $v \in V \backslash D$, there is a $z \in D$ with $d_G(v, z) \leq k$. The *k-domination number* $\gamma_k(G)$ is the minimum number of vertices in a k-dominating set. A *fractional k-domination* is a vector of non-negative weights on the vertices such that for each k-neighbourhood the weights sum up to at least one. The *fractional k-domination number* $\gamma_k^*(G)$ is the minimum sum of the weights in a fractional k-domination. Let $w : V \to \mathbb{R}_+$ be a cost function on the vertices. The *weighted k-domination number* $\gamma_k(G, w)$ is the minimum sum of the costs in a k-dominating set and the *fractional weighted k-domination number* $\gamma_k^*(G, w)$ is the minimum sum of the costs in a fractional k-domination.

Example 4.2.2 Let $G = (V, E)$ be the graph depicted in Figure 4.2, let $w_1 = (1, 1, 1, 1)$ and let $k = 1$. The minimum number of vertices in a 1-dominating set is 2. Hence, $\gamma_1(G) = \gamma_1(G, w_1) = 2$. Note that $y = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is an optimal fractional 1-domination. Therefore, $\gamma_1^*(G) = \gamma_1^*(G, w_1) = \frac{4}{3}$.

Now let $w_2 = (10, 1, 10, 1)$. Clearly, $D = \{2, 4\}$ is an optimal 1-dominating set, and $(0, 1, 0, 1)$ is an optimal fractional 1-domination. Hence, $\gamma_1(G, w_2) = \gamma_1^*(G, w_2) = 2$. ◇

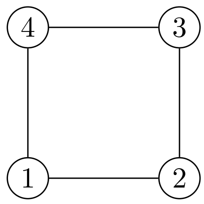

Figure 4.2: A graph $G = (V, E)$.

The *k-neighbourhood matrix* of $G = (V, E)$ is the $|V| \times |V|$-matrix $A_k(G)$, defined by $(A_k(G))_{vw} = 1$ if $w \in N_k(v)$ and $(A_k(G))_{vw} = 0$ if $w \notin N_k(v)$. The *k-th power* of $G = (V, E)$ is the graph $G^k = (V, E^k)$, where $(v, w) \in E^k$ if and only if $d_G(v, w) \leq k$. Note that $A_k(G) = A_1(G^k)$. Observe that

$$\gamma_k(G, w) = \min\{yw : yA_k(G) \geq e(V), y \in \{0, 1\}^V\}$$

and that

$$\gamma_k^*(G, w) = \min\{yw : yA_k(G) \geq e(V), y \geq 0\},$$

Let A be a $\{0, 1\}$-matrix of size $m \times n$. Let $M = \{1, \ldots, m\}$ and $N = \{1, \ldots, n\}$. Then A is called *ideal* if each extreme point of the polyhedron $P = \{x \in \mathbb{R}^M : xA \geq e(N), x \geq 0\}$ is integer. According to Lehman (1990), A is ideal if and only if $\min\{xw : xA \geq e(N), x \geq 0\} = \min\{xw : xA \geq e(N), x \in \{0, 1\}^M\}$ for all $w \in \mathbb{R}^M_+$. Note that if $A_k(G)$ is ideal, then $\gamma_k(G, w) = \gamma_k^*(G, w)$ for every $w : V \to \mathbb{R}_+$. The matrix A is called *balanced* (cf. Berge (1972)) if it does not contain an odd sized square submatrix with exactly two non-zero entries in each row and each column. If A is balanced, then all extreme points of $\{y \in \mathbb{R}^N : Ay \leq e(M), y \geq 0\}$ are integer. Furthermore, if A is balanced, then it is ideal (cf. Fulkerson, Hoffman, and Oppenheim (1974)).

We conclude this section with the definition of combinatorial optimisation games. Let N and M be finite sets, A a $\{0, 1\}$-matrix with its column set indexed by N, its row set indexed by M, each column containing at least one non-zero entry, and $w \in \mathbb{R}^M_+$. The *combinatorial optimisation game* (N, c) associated with A and w, as introduced in Deng, Ibaraki, and

Nagamochi (1999), is defined by

$$c(S) = \min\{yw : yA \geq e(S), y \in \{0,1\}^M\},$$

for each $S \subseteq N$. We remark that throughout this thesis we slightly abuse notation by omitting all transpose signs. The following two theorems are due to Deng, Ibaraki, and Nagamochi (1999).

Theorem 4.2.1 (Deng, Ibaraki, and Nagamochi (1999)) Let N and M be finite sets, A a $\{0,1\}$-matrix with its column set indexed by N, its row set indexed by M, each column containing at least one non-zero entry, $w \in \mathbb{R}_+^M$, and (N, c) its associated combinatorial optimisation game. Then $z \in C(c)$ if and only if $z \geq 0$, $Az \leq w$, and $\sum_{i \in N} z_i = c(N)$.

Theorem 4.2.2 (Deng, Ibaraki, and Nagamochi (1999)) Let N and M be finite sets, A a $\{0,1\}$-matrix with its column set indexed by N, its row set indexed by M, each column containing at least one non-zero entry, $w \in \mathbb{R}_+^M$, and (N, c) its associated combinatorial optimisation game. Then $C(c) \neq \emptyset$ if and only if

$$\min\{yw : yA \geq e(N), y \in \{0,1\}^M\} = \min\{yw : yA \geq e(N), y \geq 0\}.$$

In such a case, $z \in \mathbb{R}_+^N$ is in the core if and only if it is an optimal solution of the dual of $\min\{yw : yA \geq e(N), y \geq 0\}$.

4.3 Dominating set games

In this section we introduce three cooperative dominating set games that model the cost allocation problem arising from domination problems on graphs. Throughout the remainder of this chapter we assume that graphs are connected. For disconnected graphs the cost allocation problem can be analysed for each of its components.

Let $G = (V, E)$ be a graph, $k \in \mathbb{N}$ and $w : V \to \mathbb{R}_+$. The *relaxed dominating set game* (N, cv_k^w) allows coalitions to place facilities in every vertex present in the graph. Furthermore, coalitions are allowed to use every edge present in the graph. Formally, (N, cv_k^w) is defined by $N =$

V and $cv_k^w(S) = \min\{yw : yA_k(G) \geq e(S), y \in \{0,1\}^N\}$. Observe that $cv_k^w(S) \leq cv_k^w(T)$ for all $S \subseteq T$. Hence, (N, cv_k^w) is a monotone game. We remark that relaxed dominating set games are included in the class of combinatorial optimisation games.

In the *intermediate dominating set game* (N, ce_k^w) coalitions are not allowed to place facilities in vertices corresponding to non-members. However, coalitions are allowed to use any edge present in the graph. Formally, (N, ce_k^w) is defined by $N = V$ and $ce_k^w(S) = \min\{yw : yA_k(G) \geq e(S), y_i = 0 \text{ if } i \notin S, y \in \{0,1\}^N\}$.

The corresponding *rigid dominating set game* (N, c_k^w) is defined by $N = V$ and $c_k^w(S) = \gamma_k(G_S, w_S) = \min\{yw_S : yA_k(G_S) \geq e(S), y \in \{0,1\}^S\}$, where w_S is w restricted to S. So the cost of a coalition is equal to the minimum weighted k-domination number of the subgraph induced by this coalition. Obviously, the rigid dominating set game does not allow coalitions to place facilities in vertices corresponding to non-members. Furthermore, coalitions are only allowed to use those edges with both endpoints being member of the coalition in order to meet the proximity condition.

Example 4.3.1 Let $G = (V, E)$ be the graph depicted in Figure 4.1, $k = 2$ and $w = (3, 1, 2)$. Then $c_2^w(\{1,3\}) = 5$, because coalition $\{1,3\}$ cannot use any edge present in graph G. In (N, ce_2^w), coalition $\{1,3\}$ can use all edges of G. Therefore, $ce_2^w(\{1,3\}) = 2$. Finally, $cv_2^w(\{1,3\}) = 1$, because coalition $\{1,3\}$ can place a facility at the location of player 2. $\diamond$

Let $G = (V, E)$ be a graph, $k \in \mathbb{N}$ and $w : V \to \mathbb{R}_+$. Let (N, c_k^w), (N, ce_k^w) and (N, cv_k^w) be the corresponding dominating set games. Obviously, the rigid dominating set game is more restrictive for coalitions than the intermediate dominating set game. Similarly, the intermediate dominating set game is more restrictive than the relaxed dominating set game. This yields for all $S \subseteq N$ that $cv_k^w(S) \leq ce_k^w(S) \leq c_k^w(S)$. Also note that $c_k^w(N) = ce_k^w(N) = cv_k^w(N) = \gamma_k(G, w)$. Finally observe, because making use of edges with endpoints outside the coalition makes no sense if $k = 1$, that $c_1^w(S) = ce_1^w(S)$ for all $S \subseteq N$.

4.4 Cores of dominating set games

In this section we study the cores of dominating set games. We derive a relation between the cores, and we provide descriptions of these sets in terms of stars, substars and proper substars. Furthermore we derive one necessary and sufficient condition for the non-emptiness of the cores of all three dominating set games. So if the core of one dominating set game is non-empty, then the cores of the other two dominating set games are non-empty as well. Finally, we provide graphs with the property that the induced dominating set games has core elements for all cost functions $w : V \to \mathbb{R}_+$.

Let $G = (V, E)$ be a graph, $k \in \mathbb{N}$ and $w : V \to \mathbb{R}_+$. In the previous section we concluded that $cv_k^w(S) \leq ce_k^w(S) \leq c_k^w(S)$ for every $S \subseteq N$, and that $cv_k^w(N) = ce_k^w(N) = c_k^w(N)$. This implies that $C(cv_k^w) \subseteq C(ce_k^w) \subseteq C(c_k^w)$. Moreover, the upcoming theorem shows that the core of (N, cv_k^w) coincides with the non-negative part of the core of (N, c_k^w), as well as with the non-negative part of $C(ce_k^w)$.

Theorem 4.4.1 Let $G = (V, E)$ be a graph, $k \in \mathbb{N}$ and $w : V \to \mathbb{R}_+$. Let (N, c_k^w), (N, ce_k^w) and (N, cv_k^w) be the corresponding dominating set games. Then $C(cv_k^w) = C(c_k^w) \cap \mathbb{R}_+^N$, and $C(cv_k^w) = C(ce_k^w) \cap \mathbb{R}_+^N$.

Proof: We only show $C(cv_k^w) = C(c_k^w) \cap \mathbb{R}_+^N$. The proof of $C(cv_k^w) = C(ce_k^w) \cap \mathbb{R}_+^N$ runs similar.

First we show that $C(cv_k^w) \subseteq C(c_k^w) \cap \mathbb{R}_+^N$. As noted before, $C(cv_k^w) \subseteq C(c_k^w)$. Because (N, cv_k^w) is a monotone game, $x \geq 0$ for all $x \in C(cv_k^w)$. Hence, $C(cv_k^w) \subseteq C(c_k^w) \cap \mathbb{R}_+^N$.

Now we show that $C(c_k^w) \cap \mathbb{R}_+^N \subseteq C(cv_k^w)$. Let $x \in C(c_k^w) \cap \mathbb{R}_+^N$. Obviously, $\sum_{i \in N} x_i = c_k^w(N) = cv_k^w(N)$. It remains to show that $\sum_{i \in T} x_i \geq v(T)$ for each $T \subseteq N$. Let $T \subseteq N$. If $c_k^w(T) = cv_k^w(T)$, then $\sum_{i \in T} x_i \leq c_k^w(T) = cv_k^w(T)$. So assume that $c_k^w(T) > cv_k^w(T)$. Let $K \subseteq N$ be such that $T \subseteq \bigcup_{j \in K} N_k(j)$ and $\sum_{j \in K} w_j = cv_k^w(T)$. That is, for coalition T it is optimal in the relaxed dominating set game to place the facilities in the locations corresponding to K. Let $\bar{T} = \bigcup_{j \in K} N_k(j)$. It follows that

$c_k^w(\bar{T}) \leq \sum_{j\in K} w_j = cv_k^w(T)$. Therefore,

$$\sum_{i\in T} x_i \leq \sum_{i\in \bar{T}} x_i \leq c_k^w(\bar{T}) \leq cv_k^w(T).$$

The first inequality is satisfied because $x \geq 0$ and $T \subseteq \bar{T}$. The second inequality is due to $x \in C(c_k^w)$. We conclude that $\sum_{i\in T} x_i \leq cv_k^w(T)$ for every $T \subseteq N$. □

Now we provide descriptions of the cores of the dominating set games in terms of stars, substars and proper substars. The first proposition provides a description of the core of (N, cv_k^w) in terms of k-stars. This proposition can be proved using Theorem 4.2.1. However, for the sake of completeness we provide a proof as well, which is based on the proof of Theorem 4.2.1 in Deng, Ibaraki, and Nagamochi (1999).

Proposition 4.4.1 Let $G = (V, E)$ be a graph, $k \in \mathbb{N}$ and $w : V \rightarrow \mathbb{R}_+$. Let (N, cv_k^w) be the corresponding relaxed dominating set game. Then $x \in C(cv_k^w)$ if and only if $x \geq 0$, $\sum_{i\in N_k(j)} x_i \leq w_j$ for each $j \in V$, and $\sum_{i\in N} x_i = cv_k^w(N)$.

Proof: First we show the "only if"-part. Let $x \in C(cv_k^w)$. Obviously, $\sum_{i\in N} x_i = cv_k^w(N)$. Furthermore, $cv_k^w(N_k(j)) \leq w_j$ for each $j \in V$ implies that $\sum_{i\in N_k(j)} x_i \leq cv_k^w(N_k(j)) \leq w_j$ for each $j \in V$. Finally, monotony of (N, cv_k^w) implies that $x \geq 0$.

It remains to show the "if"-part. Let $x \geq 0$ be such that $\sum_{i\in N_k(j)} x_i \leq w_j$ for each $j \in V$, and $\sum_{i\in N} x_i = cv_k^w(N)$. It is sufficient to show for each $T \subseteq N$ that $\sum_{i\in T} x_i \leq cv_k^w(T)$. Let $T \subseteq N$ and let $K \subseteq N$ be a set of vertices that minimises the cost of placing the facilities for coalition T, i.e. $K \subseteq N$ is such that $T \subseteq \bigcup_{j\in K} N_k(j)$ and $\sum_{j\in K} w_j = cv_k^w(T)$. Then $\sum_{i\in T} x_i \leq \sum_{j\in K} \sum_{i\in N_k(j)} x_i \leq \sum_{j\in K} w_j = cv_k^w(T)$. The first inequality is satisfied because $T \subseteq \bigcup_{j\in K} N_k(j)$ and $x \geq 0$. The second inequality is satisfied because we assumed that $\sum_{i\in N_k(j)} x_i \leq w_j$ for each $j \in V$. □

Proposition 4.4.1 provides a description of the core of relaxed dominating set games in terms of k-stars. Similarly, the next proposition provides a core description for intermediate dominating set games in terms of k-substars. In particular, an efficient cost allocation vector is a core element of an intermediate dominating set game if and only if no coalition corresponding to a k-substar has an incentive to leave the grand coalition.

Proposition 4.4.2 Let $G = (V, E)$ be a graph, $k \in \mathbb{N}$ and $w : V \to \mathbb{R}_+$. Let (N, ce_k^w) be the corresponding intermediate dominating set game. Then $x \in C(ce_k^w)$ if and only if $\sum_{i \in S} x_i \leq w_j$ for all $j \in V$ and $S \in \mathcal{S}_k(j)$, and $\sum_{i \in N} x_i = ce_k^w(N)$.

Proof: First we show the "only if" part. Let $x \in C(ce_k^w)$, $j \in V$, and $S \in \mathcal{S}_k(j)$. From the definition of k-substars it follows that $j \in S$. Hence, $ce_k^w(S) \leq w_j$ and we conclude that $\sum_{i \in S} x_i \leq ce_k^w(S) \leq w_j$. Trivially, $\sum_{i \in N} x_i = ce_k^w(N)$.

Now we show the "if" part. Let $x \in \mathbb{R}^N$ be such that $\sum_{i \in S} x_i \leq w_j$ for all $j \in V$ and $S \in \mathcal{S}_k(j)$, and $\sum_{i \in N} x_i = ce_k^w(N)$. Let $T \subseteq N$ and let $K \subseteq T$ be an optimal weighted k-dominating set of T, i.e. $T \subseteq \bigcup_{j \in K} N_k(j)$ and $\sum_{j \in K} w_j = ce_k^w(T)$. There exist disjoint k-substars $S_j \in \mathcal{S}_k(j)$, $j \in K$, such that $\bigcup_{j \in K} S_j = T$. It follows that $\sum_{i \in T} x_i = \sum_{j \in K} \sum_{i \in S_j} x_i \leq \sum_{j \in K} w_j = ce_k^w(T)$. Therefore, $x \in C(ce_k^w)$. $\square$

Finally we consider the cores of rigid dominating set games. In Proposition 4.4.3 we provide a description of the core of these games in terms of proper k-substars. In particular, an efficient cost allocation vector is in the core if and only if no coalition corresponding to a proper k-substar has an incentive to leave the grand coalition.

Proposition 4.4.3 Let $G = (V, E)$ be a graph, $k \in \mathbb{N}$ and $w : V \to \mathbb{R}_+$. Let (N, c_k^w) be the corresponding rigid dominating set game. Then $x \in C(c_k^w)$ if and only if $\sum_{i \in S} x_i \leq w_j$, for all $j \in V$ and $S \in \mathcal{P}_k(j)$, and $\sum_{i \in N} x_i = c_k^w(N)$.

Proof: First we show the "only if" part. Let $x \in C(c_k^w)$, $j \in V$, and $S \in \mathcal{P}_k(j)$. Obviously, $c_k^w(S) \leq w_j$. Therefore $\sum_{i \in S} x_i \leq c_k^w(S) \leq w_j$. Trivially, $\sum_{i \in N} x_i = c_k^w(N)$.

Now we show the "if" part. Let $x \in \mathbb{R}^N$ be such that $\sum_{i \in S} x_i \leq w_j$ for each $j \in V$ and $S \in \mathcal{P}_k(j)$, and $\sum_{i \in N} x_i = c_k^w(N)$. Let $T \subseteq N$ and let $K \subseteq T$ be an optimal k-dominating set of G_T. Hence, $\bigcup_{j \in K} N_k^{G_T}(j) = T$ and $c_k^w(T) = \sum_{j \in K} w_j$. Write $K = \{v_1, \ldots, v_m\}$. According to Lemma 4.2.1, T can be partitioned into disjoint proper k-substars U_l at v_l, $l \in \{1, \ldots, m\}$. This implies that

$$\sum_{i \in T} x_i = \sum_{l=1}^{m} \sum_{i \in U_l} x_i \leq \sum_{l=1}^{m} w_{v_l} = c_k^w(T),$$

where the inequality holds by assumption. □

In the remainder of this section we focus on non-emptiness of the cores of dominating set games. We will provide one necessary and sufficient condition for non-emptiness of the core of all three dominating set games. In particular, dominating set games have non-empty cores if and only if the fractional weighted k-domination number equals the weighted k-domination number. Before we show our main theorem, we first state and prove two technical lemmas.

Lemma 4.4.1 Let $G = (V, E)$ be a graph and $k \in \mathbb{N}$. Let $v, w \in V$ and $S \in \mathcal{P}_k(w)$ such that $v \neq w$ and $v \in S$. Let $W = \{l \in S : d_{G_S}(l, w) = d_{G_S}(l, v) + d_{G_S}(v, w)\}$. Then, $S \backslash W \in \mathcal{P}_k(w)$.

Proof: We show the lemma by contradiction. Suppose that $S \backslash W \notin \mathcal{P}_k(w)$. Then there is a $q \in S \backslash W$ with $d_{G_{S \backslash W}}(q, w) > k \geq d_{G_S}(q, w)$, where the second inequality is satisfied because $S \in \mathcal{P}_k(w)$. Because the length of every shortest (q, w)-path in $G_{S \backslash W}$ is strictly larger than the length of every shortest (q, w)-path in G_S, it must hold that every shortest (q, w)-path in G_S uses an element $l \in W$. Let $l \in W$ be such that $d_{G_S}(q, w) = d_{G_S}(q, l) + d_{G_S}(l, w)$. Because $l \in W$ it follows by definition of W that $d_{G_S}(l, w) = d_{G_S}(l, v) + d_{G_S}(v, w)$. We conclude that $d_{G_S}(q, w) = d_{G_S}(q, l) + d_{G_S}(l, v) + d_{G_S}(v, w)$, which implies that there is a shortest (q, w)-path in G_S which uses v. Therefore $q \in W$, contradicting $q \in S \backslash W$. □

Lemma 4.4.2 Let $G = (V, E)$ be a graph and $k \in \mathbb{N}$. Let $v, w, z \in V$ be distinct and $S \in \mathcal{P}_k(w)$ be such that $v \in S$ and $z \notin S$. Let P be the vertex set of a shortest (v, z)-path. Finally, let $W = \{l \in S : d_{G_S}(l, w) = d_{G_S}(l, v) + d_{G_S}(v, w)\}$. If $S \cup P \notin \mathcal{P}_k(w)$, then $d_G(q, v) < d_G(z, v)$ for all $q \in W$.

Proof: Assume that $S \cup P \notin \mathcal{P}_k(w)$. First we show that $d_{G_{S\cup P}}(z, v) + d_{G_{S\cup P}}(v, w) > k$ by contradiction. Suppose that $d_{G_{S\cup P}}(z, v)+d_{G_{S\cup P}}(v, w) \leq k$. Then, for each $q \in P$,

$$d_{G_{S\cup P}}(q, w) \leq d_{G_{S\cup P}}(q, v)+d_{G_{S\cup P}}(v, w) \leq d_{G_{S\cup P}}(z, v)+d_{G_{S\cup P}}(v, w) \leq k.$$

The first inequality is due to the triangle inequality. The second inequality is satisfied because $q \in P$, and the last inequality holds by assumption.

Since $S \in \mathcal{P}_k(w)$ it is true that for each $q \in S$, $d_{G_{S\cup P}}(q, w) \leq d_{G_S}(q, w) \leq k$. We conclude that $S \cup P \in \mathcal{P}_k(w)$ which contradicts our assumption. Hence, $d_{G_{S\cup P}}(z, v) + d_{G_{S\cup P}}(v, w) > k$.

Because $S \in \mathcal{P}_k(w)$, we have for all $q \in W$ that $d_{G_S}(q, w) \leq k$. This implies that

$$d_{G_{S\cup P}}(z, v) + d_{G_{S\cup P}}(v, w) > k \geq d_{G_S}(q, w)$$

$$= d_{G_S}(q, v) + d_{G_S}(v, w) \geq d_{G_S}(q, v) + d_{G_{S\cup P}}(v, w).$$

The equality is satisfied because $q \in W$. So we have obtained that $d_{G_{S\cup P}}(z, v) > d_{G_S}(q, v)$. This implies that

$$d_G(q, v) \leq d_{G_S}(q, v) < d_{G_{S\cup P}}(z, v) = d_G(z, v).$$

The last equality is satisfied because P is the vertex set of a shortest (z, v)-path. □

Theorem 4.4.2 Let $G = (V, E)$ be a graph, $k \in \mathbb{N}$ and $w : V \to \mathbb{R}_+$. Let (N, cv_k^w), (N, ce_k^w) and (N, c_k^w) be the corresponding dominating set games. The following statements are equivalent:

1. $\gamma_k(G, w) = \gamma_k^*(G, w)$;

2. $C(cv_k^w) \neq \emptyset$;

3. $C(ce_k^w) \neq \emptyset$;

4. $C(c_k^w) \neq \emptyset$.

Proof: First we show that 1 and 2 are equivalent. We remark that this equivalence follows directly from Theorem 4.2.2, but for the sake of completeness we provide a proof as well.

According to Proposition 4.4.1, $C(cv_k^w) \neq \emptyset$ if and only if there is an $x \geq 0$ with $\sum_{i \in N_k(j)} x_i \leq w_j$ for each $j \in V$, and $\sum_{i \in N} x_i = cv_k^w(N)$. Now first observe that $\sum_{i \in N_k(j)} x_i \leq w_j$ for each $j \in V$ if and only if $A_k(G)x \leq w$. So the conditions $x \geq 0$, $\sum_{i \in N_k(j)} x_i \leq w_j$ for each $j \in V$, and $\sum_{i \in N} x_i = cv_k^w(N)$ are feasible if and only if

$$cv_k^w(N) \leq \max\{\sum_{i \in N} z_i : A_k(G)z \leq w, z \geq 0\}. \tag{4.1}$$

We conclude that $C(cv_k^w) \neq \emptyset$ if and only if (4.1) is satisfied. Now observe that the right-hand side of (4.1) coincides, according to Theorem 1.2.2, with

$$\min\{yw : yA_k(G) \geq e(N), y \geq 0\}.$$

We recognise this last expression as $\gamma_k^*(G, w)$. Since $cv_k^w(N) = \gamma_k(G, w)$, we conclude that $C(cv_k^w) \neq \emptyset$ if and only if $\gamma_k(G, w) \leq \gamma_k^*(G, w)$. Since $\gamma_k^*(G, w) \leq \gamma_k(G, w)$ by definition, it follows that $C(cv_k^w) \neq \emptyset$ if and only if $\gamma_k(G, w) = \gamma_k^*(G, w)$.

Implications "$2 \Rightarrow 3$" and "$3 \Rightarrow 4$" follow from the observation that $C(cv_k^w) \subseteq C(ce_k^w) \subseteq C(c_k^w)$. So it remains to show "$4 \Rightarrow 2$". We will show that if $C(c_k^w) \neq \emptyset$, then there is a $y \in C(c_k^w)$ with $y \geq 0$. This implies, according to Theorem 4.4.1, that $y \in C(cv_k^w)$.

We produce a non-negative element of $C(c_k^w)$ by means of an algorithm. This algorithm uses as input a core element of (N, c_k^w). Then, in each step, the core element is altered by raising the amount given to one player, and lowering the amount given to another player. We will argue that the resulting allocation is still a core element. Furthermore we show that, in a finite number of steps, the algorithm converges to a non-negative core element.

Algorithm: Construction of a non-negative core element of $C(c_k^w)$.

Step 1: Let $y \in C(c_k^w)$, $p = 1$ and $x^p = y$.

Step 2: If $x^p \geq 0$, then stop. Else go to step 3.

Step 3: Let $(i^p, j^p) \in \operatorname{argmin}\{d_G(i,j) : x_i^p < 0, x_j^p > 0\}$. Let $\epsilon^p = \min\{x_{j^p}^p, -x_{i^p}^p\} > 0$. Let $x_{i^p}^{p+1} = x_{i^p}^p + \epsilon^p$, $x_{j^p}^{p+1} = x_{j^p}^p - \epsilon^p$ and $x_j^{p+1} = x_j^p$ for all $j \in N \backslash \{i^p, j^p\}$. Let $p = p+1$, and return to step 2.

First we show that $x^p \in C(c_k^w)$ for all p by induction on p. Subsequently we prove that the algorithm stops after a finite number of steps, and hence converges to a non-negative core element.

Note that $x^1 \in C(c_k^w)$. As the induction hypothesis, assume that $x^p \in C(c_k^w)$. Suppose that $x^p \ngeq 0$. Let $(i^p, j^p) \in \operatorname{argmin}\{d_G(i,j) : x_i^p < 0, x_j^p > 0\}$. Let P be the vertex set of a shortest (i^p, j^p)-path. By definition of i^p and j^p, $x_l = 0$ for each $l \in P \backslash \{i^p, j^p\}$.

According to Proposition 4.4.3 it is sufficient to show for all $j \in N$ and all $S \in \mathcal{P}_k(j)$, that $\sum_{l \in S} x_l^{p+1} \leq w_j$. In fact, since $x^p \in C(c_k^w)$, it suffices to consider those proper k-substars that are allocated a larger amount at x^{p+1} than at x^p. Hence, we only need to consider those coalitions that contain i^p, but do not contain j^p. So let $j \in N$ and $S \in \mathcal{P}_k(j)$ be such that $i^p \in S$ and $j^p \notin S$. We distinguish between two cases.

`Case 1:` $S \cup P \in \mathcal{P}_k(j)$.

From $x^p \in C(c_k^w)$ and the assumption that $S \cup P \in \mathcal{P}_k(j)$, we obtain that $\sum_{l \in S \cup P} x_l^p \leq w_j$. Because $x_l^p = 0$ for all $l \in P \backslash \{i^p, j^p\}$,

$$\sum_{l \in S} x_l^p \leq w_j - x_{j^p}^p. \tag{4.2}$$

It follows that

$$\begin{aligned}\sum_{l\in S} x_l^{p+1} &= \sum_{l\in S} x_l^p - x_{i^p}^p + x_{i^p}^{p+1}\\ &\leq w_j - x_{j^p}^p - x_{i^p}^p + x_{i^p}^{p+1}\\ &= w_j - x_{j^p}^p + \epsilon^p\\ &\leq w_j.\end{aligned}$$

The first equality is satisfied because $x_l^{p+1} = x_l^p$ for each $l \in S\backslash\{i^p\}$. The first inequality follows from (4.2). The last inequality is satisfied because $\epsilon^p \leq x_{j^p}^p$.

`Case 2:` $S \cup P \notin \mathcal{P}_k(j)$.

First suppose that $j = i^p$. Because $S \cup P \notin \mathcal{P}_k(i^p)$, it follows that $d_{G_{S\cup P}}(i^p, j^p) > k$. Since $S \in \mathcal{P}_k(j)$, we have $d_{G_S}(i^p, l) \leq k$ for each $l \in S$. This yields for all $l \in S$,

$$d_G(i^p, l) \leq d_{G_S}(i^p, l) \leq k < d_{G_{S\cup P}}(i^p, j^p) = d_G(i^p, j^p).$$

The equality is satisfied because P is the vertex set of a shortest (i^p, j^p)-path. We conclude that each player in S is located closer to i^p than j^p is located to i^p. It follows by definition of i^p and j^p that $x_l^p \leq 0$ for all $l \in S\backslash\{i^p\}$. Since $x_{i^p}^{p+1} \leq 0$ it follows that $\sum_{l\in S} x_l^{p+1} \leq 0 \leq w_{i^p}$.

Secondly, suppose that $j \neq i^p$. Let $W = \{l \in S : d_{G_S}(l, j) = d_{G_S}(l, i^p) + d_{G_S}(i^p, j)\}$ be the set of vertices in S for which a shortest path to j uses i^p. From Lemma 4.4.1, with $v = i^p$ and $w = j$, it follows that $S\backslash W \in \mathcal{P}_k(j)$. According to Lemma 4.4.2, with $v = i^p$, $w = j$ and $z = j^p$, it is satisfied that $d_G(l, i^p) < d_G(j^p, i^p)$ for all $l \in W$. Using the definition of i^p and j^p it follows that $x_l^{p+1} \leq 0$ for all $l \in W$. Therefore

$$\begin{aligned}\sum_{l\in S} x_l^{p+1} &= \sum_{l\in S\backslash W} x_l^p + \sum_{l\in W, l\neq i^p} x_l^p + x_{i^p}^{p+1}\\ &\leq w_j + \sum_{l\in W, l\neq i^p} x_l^p + x_{i^p}^{p+1}\\ &\leq w_j.\end{aligned}$$

The first inequality is satisfied since $S \backslash W \in \mathcal{P}_k(j)$ and therefore, $\sum_{l \in S \backslash W} x_l^p \leq w_j$. The last inequality follows from $x_l^p \leq 0$ for all $l \in W$ and $x_{i^p}^{p+1} \leq 0$.

We conclude that $x^p \in C(c_k^w)$ for all p. It remains to show that the algorithm converges. First observe that the algorithm continues as long as there are players that are allocated a negative amount. However, note that by definition of ϵ^p, either $x_{j^p}^{p+1} = 0$ and $x_{i^p}^{p+1} \leq 0$, or $x_{j^p}^{p+1} \geq 0$ and $x_{i^p}^{p+1} = 0$. Hence, x^{p+1} contains at least one zero entry more than x^p. Because x^p is an $|N|$-dimensional vector, the algorithm produces a non-negative core element in at most $|N|$ steps. $\square$

The cores of the dominating set games are non-empty if and only if $\gamma_k(G,w) = \gamma_k^*(G,w)$. Unfortunately, the problem of determining $\gamma_k(G,w)$ is NP-complete in general. Hence, it is difficult to determine whether $\gamma_k(G,w) = \gamma_k^*(G,w)$. For some classes of graphs however, the k-domination problem is relatively easy to solve. For example, a special subclass of chordal graphs satisfies this property.

A *circuit* is a connected graph on at least three vertices such that each vertex is adjacent to precisely two other vertices. A circuit on n vertices is denoted by C_n. A graph is called *chordal* if it does not contain a circuit of length at least four as an induced subgraph. A *sun* is a chordal graph on $2n$ vertices for some $n \geq 3$, whose vertex set can be partitioned into two sets, $W = \{w_1, \ldots, w_n\}$ and $U = \{u_1, \ldots, u_n\}$ such that any two vertices of W are non-adjacent, and for each $i, j \in \{1, \ldots, n\}$, w_i is adjacent to u_j if and only if $i = j$ or $i = j+1$ $(mod\ n)$. A graph is called an *odd (even) sun* if it is a sun on $2n$ vertices, with n odd (even). An *(odd-)sun-free chordal* graph is a chordal graph which does not contain an (odd-)sun as an induced subgraph. Sun-free chordal graphs are called *strongly chordal* graphs in Farber (1981). The concept of an even sun is illustrated in the following example.

Example 4.4.1 Let $G = (V, E)$ be the graph depicted in Figure 4.3. Observe that G is chordal. Moreover, the sets $W = \{w_1, \ldots, w_6\}$ and $U = \{u_1, \ldots, u_6\}$ form a partition of V. Any two vertices of W are non-adjacent, and, w_i is connected to u_j if and only if $i = j$ or $i = j + 1$ $(mod\ 6)$. Hence, G is a sun. Because $|U| = |W| = 6$ we conclude that G is an even sun. $\diamond$

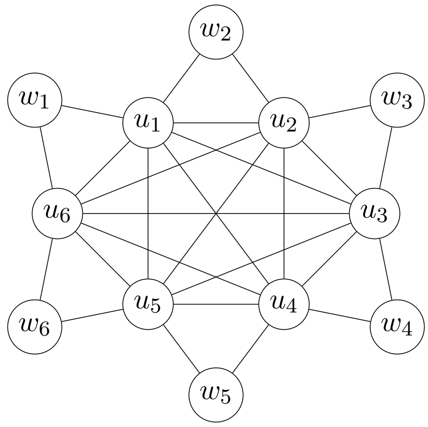

Figure 4.3: An even sun.

If a graph is sun-free chordal, then it is necessarily odd-sun-free chordal. Trees, line graphs of trees, interval graphs and block graphs are examples of sun-free chordal graphs (cf. Farber (1981)). It can be determined in polynomial time whether a graph is sun-free chordal (cf. Farber (1981)). Theorem 4.4.3 states that odd-sun-free chordal graphs are characterised by balancedness of their 1-neighbourhood matrices.

Theorem 4.4.3 (Brouwer, Duchet, and Schrijver (1983)) Let G be a graph. Then $A_1(G)$ is balanced if and only if G is odd-sun-free chordal.

Lubiw (1982) showed that powers of sun-free chordal graphs are sun-free chordal as well. Hence, if $G = (V, E)$ is a sun-free chordal graph, then $A_k(G) = A_1(G^k)$ is balanced for all $k \in \mathbb{N}$. This implies that $\gamma_k(G, w) = \gamma_k^*(G, w)$ for all $w : V \to \mathbb{R}_+$ and all $k \in \mathbb{N}$. Straightforwardly we have the following proposition.

Proposition 4.4.4 Let $G = (V, E)$ be sun-free chordal. Then the corresponding dominating set games have core elements for all $k \in \mathbb{N}$ and for all $w : V \to \mathbb{R}_+$.

Powers of odd-sun-free chordal graphs are not necessarily odd-sun-free chordal. For example, let $G = (V, E)$ be the 6-sun depicted in Figure 4.3. Then the subgraph of G^2 induced by $\{w_1, \ldots, w_6\}$ is a circuit on 6 vertices. This implies that G^2 is not chordal and therefore also not odd-sun-free chordal.

Obviously, circuits are not chordal, and therefore not sun-free chordal. However, Cornuéjols and Novick (1994) showed that $A_1(C_6)$ and $A_1(C_9)$ are ideal matrices. Moreover, they showed that $A_1(C_6)$ and $A_1(C_9)$ are the only ideal matrices of the form $A_k(C_n)$ with $k, n \in \mathbb{N}$ such that $k \leq \frac{n-2}{2}$. Note that if $k > \frac{n-2}{2}$, then $A_k(C_n)$ is the matrix with every entry a one.

Theorem 4.4.4 (Cornuéjols and Novick (1994)) The matrices $A_1(C_6)$ and $A_1(C_9)$ are ideal.

From Theorem 4.4.4 it follows that $\gamma_1(C_6, w) = \gamma_1^*(C_6, w)$ and $\gamma_1(C_9, w) = \gamma_1^*(C_9, w)$ for every $w : V \to \mathbb{R}_+$. Hence, we have the following proposition.

Proposition 4.4.5 Let $G = C_6$ or $G = C_9$. Then the corresponding dominating set games have core elements for $k = 1$ and for all $w : V \to \mathbb{R}_+$.

We conclude this section with the observation that a special subclass of relaxed dominating set games satisfies the CoMa-property. A game $v \in TU^N$ with $C(v) \neq \emptyset$ is said to satisfy the *CoMa-property* if all extreme points of the core are marginal vectors. Obviously, all concave games satisfy the CoMa-property, but the contrary does not necessarily hold (cf. Kuipers (1993) and Hamers, Klijn, Solymosi, Tijs, and Villar (2002)).

The next theorem shows that a special subclass of combinatorial optimisation games satisfies the CoMa-property. We then apply this theorem to the class of relaxed dominating set games, since relaxed dominating set games form a subclass of combinatorial optimisation games.

Proposition 4.4.6 Let A be a $\{0, 1\}$-matrix of size $m \times n$. Let $w_i = 1$ for each $i \in \{1, \ldots, m\}$. If A is balanced, then the combinatorial optimisation game (N, c) associated with A and w satisfies the CoMa-property.

Proof: Since A is balanced it follows that A is ideal. This implies that each extreme point of the polyhedron $\{x \in \mathbb{R}^M : xA \geq e(N), x \geq 0\}$ is integer and therefore that $\min\{xw : xA \geq e(N), x \geq 0\} = \min\{xw : xA \geq$

$e(N), x \in \{0,1\}^M\}$. According to Theorem 4.2.2 it follows that $C(c) \neq \emptyset$. It remains to show that each extreme point of $C(c)$ corresponds to a marginal vector. First we show that each extreme point of $C(c)$ is a $\{0,1\}$-vector, and then we argue that this vector is indeed a marginal vector.

Let $M = \{1,\ldots,m\}$ and $N = \{1,\ldots,n\}$. Since each extreme point of the polyhedron $\{x \in \mathbb{R}^M : xA \geq e(N), x \geq 0\}$ is integer, it follows that

$$\begin{aligned} c(N) &= \min\{\sum_{i\in M} x_i : xA \geq e(N), x \in \{0,1\}^M\} \\ &= \min\{\sum_{i\in M} x_i : xA \geq e(N), x \geq 0\} \\ &= \max\{\sum_{i\in N} y_i : Ay \leq e(M), y \geq 0\}. \end{aligned} \tag{4.3}$$

The third equality is satisfied due to Theorem 1.2.2. According to Theorem 4.2.2, it holds that $y \in C(c)$ if and only if y is an optimal solution of (4.3). From the balancedness of A we conclude that each extreme point of $\{y \in \mathbb{R}^N : Ay \leq e(M), y \geq 0\}$ is integer (cf. Berge (1972)). Because the set of optimal solutions of (4.3) is a facet of $\{y \in \mathbb{R}^N : Ay \leq e(N), y \geq 0\}$, we conclude that each extreme point this facet, and hence each extreme point of $C(c)$, is integer. In particular, each extreme point of $C(c)$ is a $\{0,1\}$-vector.

Now let $x \in C(c)$ be an extreme point and let $S = \{i \in N : x_i = 1\}$. Let $\sigma \in \Pi(N)$ be such that $\sigma(i) \in S$ for each $i \in \{1,\ldots,|S|\}$. That is, σ begins with the members of S, and ends with the members of $N \backslash S$. We show that $m^\sigma(c) = x$.

Let $i \in \{1,\ldots,|S|\}$. Because $x \in C(c)$, it follows that $|[\sigma(i),\sigma]| = \sum_{j=1}^{i} x_{\sigma(j)}$
$\leq c([\sigma(i),\sigma])$. By definition of (N,c), $c(T) \leq |T|$ for each $T \subseteq N$. We conclude that $c([\sigma(i),\sigma]) = |[\sigma(i),\sigma]|$ for each $i \in \{1,\ldots,|S|\}$. This implies that

$$x_{\sigma(i)} = 1 = c([\sigma(i),\sigma]) - c([\sigma(i-1),\sigma]) = m^\sigma_{\sigma(i)}(c).$$

Since $c(S) = \sum_{i\in S} x_i = \sum_{i\in N} x_i = c(N)$, it follows from monotony of (N,c) that $c(S) = c(T) = c(N)$ for each $S \subseteq T \subseteq N$. This implies that

$$x_{\sigma(i)} = 0 = c([\sigma(i),\sigma]) - c([\sigma(i-1),\sigma]) = m^\sigma_{\sigma(i)}(c),$$

for each $i \in \{|S|+1, \ldots, |N|\}$. We conclude that $x = m^\sigma(c)$. □

We immediately have the following corollary.

Corollary 4.4.1 Let $G = (V, E)$ be a graph, $k \in \mathbb{N}$ and $w_i = 1$ for each $i \in V$. If G^k is odd-sun-free chordal, then (N, cv_k^w) satisfies the CoMa-property.

Unfortunately, Corollary 4.4.1 does not extend to arbitrary cost functions. This is illustrated in the following example.

Example 4.4.2 Let $G = (V, E)$ be the graph depicted in Figure 4.4 and let $k = 1$. Clearly, G is a tree, and therefore odd-sun-free chordal graph. Let $w = (6, 4, 4, 4)$. Then,

$$cv_1^w(S) = \begin{cases} 4, & \text{if } |S \cap \{2,3,4\}| \leq 1,\ S \neq \emptyset; \\ 6, & \text{if } |S \cap \{2,3,4\}| \geq 2. \end{cases}$$

It is straightforward to check that $(3, 1, 1, 1)$ is an extreme point of $C(cv_1^w)$. However, x does not correspond to a marginal vector. ◇

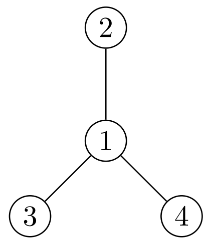

Figure 4.4: An odd-sun-free chordal graph.

4.5 Concavity

In this section we consider concavity of dominating set games. We characterise concavity of dominating set games in terms of the underlying graph if $w_i = 1$ for all $i \in V$. First we show that a relaxed dominating set game with parameter k is concave if and only if the corresponding graph contains a vertex with distance at most k to all other vertices.

Proposition 4.5.1 Let $G = (V, E)$ be a graph, $k \in \mathbb{N}$ and $w_i = 1$ for all $i \in V$. Let (N, cv_k^w) be the corresponding relaxed dominating set game. Then (N, cv_k^w) is concave if and only if $k \geq r(G)$.

Proof: Note that $k \geq r(G)$ if and only if there is a $v \in V$ with $N_k(v) = V$. First we show the sufficiency part. Assume there is a $v \in V$ with $N_k(v) = V$. Then $cv_k^w(S) = 1$ for all $S \subseteq N$, $S \neq \emptyset$. Hence, (N, cv_k^w) is concave.

Now we prove necessity. Assume that $N_k(v) \neq V$ for all $v \in V$. We show that (N, cv_k^w) is not concave. Hence, the condition that there is a $v \in V$ with $N_k(v) = V$ is necessary for concavity.

Let $v \in V$ be such that $N_k(v)$ is a maximal k-neighbourhood in the sense that it is not a proper subset of any other k-neighbourhood. By assumption, $N_k(v) \neq V$. Now let $u \in V \backslash N_k(v)$ be such that $d_G(v, u) = k + 1$. Obviously, $N_k(v) \cap N_k(u) \neq \emptyset$, $cv_k^w(N_k(v)) = 1$, $cv_k^w(N_k(u)) = 1$ and $cv_k^w(N_k(v) \cap N_k(u)) = 1$. Because $N_k(v)$ is a maximal k-neighbourhood and $u \notin N_k(v)$, we conclude that $(N_k(v) \cup N_k(u)) \not\subseteq N_k(y)$ for each $y \in V$. Thus, $cv_k^w(N_k(v) \cup N_k(u)) = 2$. Therefore $cv_k^w(N_k(v) \cap N_k(u)) + cv_k^w(N_k(v) \cup N_k(u)) = 3 > 2 = cv_k^w(N_k(v)) + cv_k^w(N_k(u))$ and we conclude that (N, cv_k^w) is not concave. $\square$

Next we consider concavity of rigid dominating set games. Before we characterise concavity for these games in case $w_i = 1$ for all $i \in V$, we introduce the concept of block graphs.

A vertex is called a *cutvertex* if the subgraph $(V \backslash \{v\}, E_{V \backslash \{v\}})$ consists of more components than G. A *bridge* is an edge $e \in E$ with the same property, i.e. if $(V, E \backslash \{e\})$ has more components than G. A graph with at least three vertices is called *2-connected* if it does not contain a cutvertex. A subgraph B is called a *block* if it is a bridge or a maximal 2-connected subgraph. A connected graph is a *block graph* if every block is complete. Note for example that a tree is a block graph. The concept of block graphs is illustrated in the following example.

Example 4.5.1 Let $G = (V, E)$ be the graph depicted in Figure 4.5. The vertices $5, 6$ and 9 are cutvertices, and the edge $\{5, 6\}$ is a bridge. The blocks are $\{1,2,3,4,5\}$, $\{5,6\}$, $\{6,7,8,9\}$ and $\{9,10,11\}$. Because each block is complete, G is a block graph. $\diamond$

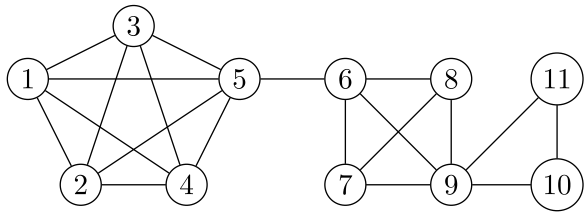

Figure 4.5: A block graph.

The following lemma that is used in the proof of our characterisation of concave rigid dominating set games, provides a relation between the radius and the diameter of a block graph.

Lemma 4.5.1 Let $G = (V, E)$ be a block graph and $k \in \mathbb{N}$. If $\Delta(G) \leq 2k$, then $r(G) \leq k$.

Proof: Block graphs are 3-sun-free chordal graphs. For 3-sun-free chordal graphs, $r(G) = \lceil \frac{\Delta(G)}{2} \rceil$ (cf. Theorem 3.6 in Chang and Nemhauser (1984)). Hence, if G is a block graph satisfying $\Delta(G) \leq 2k$, then $r(G) \leq k$. $\square$

Proposition 4.5.2 Let $G = (V, E)$ be a graph, $k \in \mathbb{N}$ and $w_i = 1$ for all $i \in V$. Let (N, c_k^w) be the corresponding rigid dominating set game. Then (N, c_k^w) is concave if and only if G is a block graph satisfying $\Delta(G) \leq 2k$.

Proof: First we show the "only if" part. Suppose that G does not satisfy $\Delta(G) \leq 2k$. Let $v, u \in V$ be such that $d_G(v, u) = 2k+1$. Let P be the vertex set of a shortest (v, u)-path. Observe that $c_k^w(P\backslash\{v, u\}) = c_k^w(P\backslash\{u\}) = c_k^w(P\backslash\{v\}) = 1$ and $c_k^w(P) = 2$. Therefore, $c_k^w(P\backslash\{v, u\}) + c_k^w(P) = 3 > 2 = c_k^w(P\backslash\{u\}) + c_k^w(P\backslash\{v\})$. We conclude that (N, c_k^w) is not concave.

Now suppose that G is not a block graph. Then there is an incomplete block. Hence, G contains C_m with $m \geq 4$ or the graph depicted in Figure 4.6 as an induced subgraph. We distinguish between three cases.

`Case 1:` G contains C_m as an induced subgraph with $m \in \{4, \ldots, 2k+2\}$.

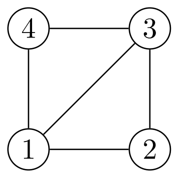

Figure 4.6: A possible subgraph of an incomplete block.

Let u and v be non-adjacent vertices in C_m. Obviously, C_m contains two disjoint paths connecting u and v. Let P_1 and P_2 denote the vertex sets of these paths. Because $\{u,v\} \notin E$ it follows that $P_1 \backslash \{u,v\} \neq \emptyset$ and $P_2 \backslash \{u,v\} \neq \emptyset$. Because u and v are non-adjacent, $c_k^w(\{u,v\}) = 2$. Furthermore, $c_k^w(P_1) = 1$, $c_k^w(P_2) = 1$ and $c_k^w(P_1 \cup P_2) \geq 1$. Thus, $c_k^w(\{u,v\}) + c_k^w(P_1 \cup P_2) \geq 3 > 2 = c_k^w(P_1) + c_k^w(P_2)$. Hence, (N, c_k^w) is not concave.

`Case 2:` G contains C_m as an induced subgraph with $m > 2k+2$.

Let u, v, z be vertices in C_m with u and v adjacent, as well as v and z. Let $H = C_m \backslash \{v\}$ be the induced subgraph of C_m obtained by deleting v. Obviously, H is an induced subgraph of G satisfying $d_H(u,z) \geq 2k+1$. It is now straightforward to show, similar to the first part of this proof, that the subgame of (N, c_k^w) associated with coalition H is not concave. This implies that (N, c_k^w) is not concave as well.

`Case 3:` G contains the graph depicted in Figure 4.6 as an induced subgraph.

It is clear that $c_k^w(\{2,4\}) = 2$, $c_k^w(\{1,2,4\}) = 1$, $c_k^w(\{2,3,4\}) = 1$ and $c_k^w(\{1,2,3,4\}) = 1$. Hence, $c_k^w(\{2,4\}) + c_k^w(\{1,2,3,4\}) = 3 > 2 = c_k^w(\{1,2,4\}) + c_k^w(\{2,3,4\})$. Therefore, (N, c_k^w) is not concave.

It remains to show the "if" part. Assume that $G = (V, E)$ is a block graph satisfying $\Delta(G) \leq 2k$. We will show that the corresponding rigid dominating set game is concave. First we show that the cost of each connected coalition is equal to 1.

Let $T \subseteq N$ be such that G_T is connected. Then G_T is again a block graph satisfying $\Delta(G_T) \leq 2k$. It follows from Lemma 4.5.1 that $r(G_T) \leq k$.

So there is a $v \in T$ with $d_{G_T}(v,u) \leq k$ for every $u \in T$. As a result, $c_k^w(T) = 1$.

Now let $i,j \in N$, $i \neq j$, and $S \subseteq N \backslash \{i,j\}$. We will show that $c_k^w(S \cup \{i,j\}) + c_k^w(S) \leq c_k^w(S \cup \{i\}) + c_k^w(S \cup \{j\})$. Denote the maximally connected components of S by $S_1, \ldots, S_p$. Since the cost of each component is equal to 1, $c_k^w(S) = p$. Let $I \subseteq \{1, \ldots, p\}$ be the index set of the components that are connected with i. That is, for all $l \in I$ there is a $v \in S_l$ with $\{v,i\} \in E$. Note that $c_k^w(S \cup \{i\}) = p + 1 - |I|$. Similarly, let $J \subseteq \{1, \ldots, p\}$ be the index set of the components that are connected with j and note that $c_k^w(S \cup \{j\}) = p + 1 - |J|$.

First we show that $|I \cap J| \leq 1$ by contradiction. Suppose that $|I \cap J| \geq 2$. Then at least two components, say S_1 and S_2, are connected with both i and j. Let $m_1, m_2 \in S_1$ be such that $\{m_1, i\}, \{m_2, j\} \in E$. Similarly, let $m_3, m_4 \in S_2$ be such that $\{m_3, i\}, \{m_4, j\} \in E$. Let $P_1 \subseteq S_1$ be the set of vertices corresponding to a shortest (m_1, m_2)-path in G_{S_1}, and let $P_2 \subseteq S_2$ be the set of vertices corresponding to a shortest (m_3, m_4)-path in G_{S_2}. The subgraph induced by $P_1 \cup P_2 \cup \{i,j\}$ forms a circuit, and hence a 2-connected subgraph. Because G is a block graph, it follows that this subgraph is complete. This implies that $\{m_1, m_3\} \in E$, contradicting that S_1 and S_2 are disconnected components. We conclude that $|I \cap J| \leq 1$.

If $|I \cap J| = 0$ and $\{i,j\} \notin E$ or if $|I \cap J| = 1$ and $\{i,j\} \in E$, then $c_k^w(S \cup \{i,j\}) = p + 2 - |I| - |J|$. If $|I \cap J| = 0$ and $\{i,j\} \in E$ or if $|I \cap J| = 1$ and $\{i,j\} \notin E$, then $c_k^w(S \cup \{i,j\}) = p + 1 - |I| - |J|$. In either case, $c_k^w(S \cup \{i,j\}) \leq p + 2 - |I| - |J|$. Therefore, $c_k^w(S \cup \{i\}) + c_k^w(S \cup \{j\}) = 2p + 2 - |I| - |J| \geq c_k^w(S) + c_k^w(S \cup \{i,j\})$. □

The final part of this section is dedicated to concavity of intermediate dominating set games. Let $G = (V,E)$ be a graph, and let $w_i = 1$ for all $i \in V$. First note that for $k = 1$ the corresponding intermediate dominating set game coincides with the rigid dominating set game. From Proposition 4.5.2 we obtain that (N, ce_1^w) is concave if and only if G is a block graph satisfying $\Delta(G) \leq 2$. For $k \geq 2$ the intermediate dominating set game does not necessarily coincide with the rigid dominating set game. The characterisations of concavity do also not coincide. In fact, the following proposition shows that intermediate dominating set games with parameter k are concave if and only if the diameter of G is at most k.

Proposition 4.5.3 Let $G = (V, E)$ be a graph, $k \in \mathbb{N}$ with $k \geq 2$, and $w_i = 1$ for all $i \in V$. Let (N, ce_k^w) be the corresponding intermediate dominating set game. Then (N, ce_k^w) is concave if and only if $\Delta(G) \leq k$.

Proof: First note that $\Delta(G) \leq k$ if and only if $N_k(v) = V$ for all $v \in V$. First we show the "if" part. Assume that $N_k(v) = V$ for all $v \in V$. Then, $ce_k^w(S) = 1$ for all $S \subseteq N$. Trivially, (N, ce_k^w) is concave.

It remains to show the "only if" part. Assume that there is a $v \in V$ with $N_k(v) \neq V$. Then there is a $u \in V$ with $d_G(v, u) = k + 1$. Let P be the vertex set of a shortest (v, u)-path. Let $a, b \in P$ be such that $\{a, v\} \in E$ and $\{b, u\} \in E$. Since $k \geq 2$, the shortest path between v and u contains at least 4 vertices. Therefore $a \neq b$. Now note that $ce_k^w(\{v, u\}) = 2$, $ce_k^w(\{v, a, u\}) = 1$, $ce_k^w(\{v, b, u\}) = 1$ and $ce_k^w(\{v, a, b, u\}) = 1$. It follows that $ce_k^w(\{v, a, b, u\}) + ce_k^w(\{v, u\}) = 3 > 2 = ce_k^w(\{v, a, u\}) + ce_k^w(\{v, b, u\})$.
□

Chapter 5

Fixed tree games with multi-located players

5.1 Introduction

In this chapter we consider a generalisation of the fixed tree problem, introduced by Megiddo (1978). In a fixed tree problem a rooted tree Γ and a set of agents N is given, each agent being located at precisely one vertex of Γ and each vertex containing precisely one agent. Megiddo (1978) associates to such a problem a cooperative cost game (N, c), a *fixed tree game*, where $c(S)$ denotes the minimal cost needed to connect all members of S to the root via a subtree of Γ, for every coalition $S \subseteq N$.

Fixed tree games and variants of fixed tree games have also been studied in Galil (1980), Granot, Maschler, Owen, and Zhu (1996), Koster, Molina, Sprumont, and Tijs (2001) and Maschler, Potters, and Reijnierse (1995). The special case where the tree is a chain corresponds to airport games, which have been considered in Littlechild (1974), Littlechild and Owen (1977) and Littlechild and Thompson (1977). Variants of fixed tree games, where it is allowed that one vertex is occupied by more players or by no player, are considered in e.g. Koster (1999) and Van Gellekom (2000). However, these variants still require that every player is located in precisely one vertex.

In this chapter, which is based on Miquel, Van Velzen, Hamers, and

Norde (2003), we generalise the model of Megiddo (1978) in the sense that players may be located in more than one vertex of the given tree. As a motivation for this generalised model one can consider the following irrigation problem. Consider a set of parcels in a desert environment which need to be irrigated from a well. For that reason, a network has been designed which allows the transportation of water from the well to at least one of the corners of each parcel. Consider, for example, the situation with five parcels in Figure 5.1.

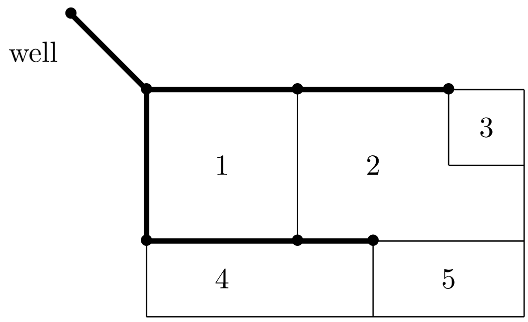

Figure 5.1: An irrigation network.

Bold lines indicate the network that has already been constructed. The players are facing the problem of dividing the maintenance costs of this network, so they are facing a fixed tree problem where the tree is as depicted in Figure 5.2.

With this fixed tree we can associate in a natural way a cooperative cost game (N, c). For every coalition $S \subseteq N$, $c(S)$ denotes the minimal cost needed to connect all members of S at least once to the root via a subtree. This leads to the class of *fixed tree games with multi-located players*.

Standard fixed tree games and the variants of these games are known to be concave. We will show that this needs not be true for fixed tree games with multi-located players. However, we will show that these games have non-empty cores by showing that the core of a fixed tree game with multi-

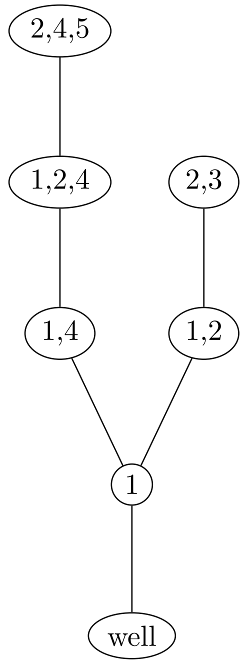

Figure 5.2: Fixed tree of the problem in Figure 5.1.

located players coincides with the core of a related standard fixed tree game. Furthermore we analyse which marginal vectors provide core elements and conclude that only standard fixed tree games are concave. We show that the Shapley value need not be a core element and we study the average of the extreme points of the core.

The remainder of this chapter is organised as follows. In Section 5.2 we formally introduce fixed tree games with multi-located players and in Section 5.3 we focus on the structure of the core of these games. The last section, Section 5.4, is dedicated to three one-point solution concepts.

5.2 Fixed tree problems with multi-located players and games

In this section we introduce fixed tree problems with multi-located players and its associated cooperative games. First we introduce some notation.

A tree (V, E) is called rooted in case V contains a special element referred to as the root. For each $v \in V$ there is a unique path from the root to v. We denote the vertex set of this path by $P(v)$. A *trunk* of (V, E) is a set of vertices $T \subseteq V$ such that $P(v) \subseteq T$ for each $v \in T$. The set of *followers* of a vertex v is the set $F(v) = \{v' \in V | v \in P(v')\}$. A vertex v is called a *leaf*[4] if $F(v) = \{v\}$. Analogously we define the set of edges $F(e)$ following an edge e. Note that $e \in F(e)$.

Now we introduce fixed tree problems with multi-located players. A *fixed tree problem with multi-located players*, FMP problem for short, is a 5-tuple $\Gamma = (N, (V, E), 0, S, a)$, where

1. N is a finite set of players;
2. (V, E) is a tree with vertex set V and edge set E;
3. 0 is a special element of V, called the root of the tree;
4. $S : V \to 2^N$ is a map assigning to each vertex a (possibly empty) subset of players;
5. $a : E \to \mathbb{R}_{++}$ is a map expressing the maintenance cost of each edge;

and which satisfies the following assumptions:

(A1) for every $i \in N$ there is a $v \in V$ with $i \in S(v)$;

(A2) for each leaf $t \in V$, there is an $i \in N$ such that $i \in S(t)$ and $i \notin S(v)$ for every $v \in V \backslash \{t\}$.

Assumption (A1) states that every player should occupy at least one vertex in the tree. Assumption (A2) states that the tree (V, E) is "optimal" for the grand coalition N, in the sense that no proper subtree of (V, E) provides at least one connection to the root for every $i \in N$.

Players who occupy at least two vertices are called *multi-located players*, and players who occupy precisely one vertex are called *single-located*. If

[4]We remark that this definition of leaf is not consistent with its definition in Section 1.2.4. The difference is that the definition in Section 1.2.4 allows the root to be a leaf, while the definition in this section forbids this possibility.

an FMP problem does not contain multi-located players, then it is called a *standard FMP* problem.

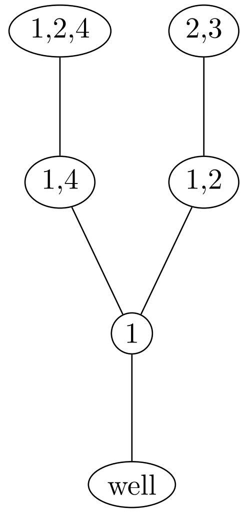

Figure 5.3: A tree depicting a cost sharing problem.

Example 5.2.1 In Figures 5.2 and 5.3 two cost sharing problems are depicted. We can easily see that the tree of Figure 5.3 does not correspond to an FMP problem because after removing the edge between the vertex occupied by players 1 and 4, and the vertex occupied by players 1, 2 and 4, all players remain connected to the root. Hence, Assumption (A2) is violated.

The tree of Figure 5.2, on the contrary, corresponds to an FMP problem. It has two leaves and each of them is occupied by one single-located player. Nevertheless, it is not a standard tree problem as defined in Megiddo (1978) since players 1, 2 and 4 are located in more than one vertex. $\diamond$

For an FMP problem we define the associated cost game as follows. Let $\Gamma = (N, (V, E), 0, S, a)$ be an FMP problem. The associated *fixed tree game with multi-located players*, FMP game for short, is the cost game (N, c) defined by

$$c(S) = \min_{T_S \in A_S} \left(\sum_{e \in T_S} a(e) \right),$$

for each $S \subseteq N$, where A_S is the collection of admissible subtrees for coalition S. A subtree is admissible for coalition $S \subseteq N$ if it provides at least one connection to the root for every member of S. In the following example we illustrate the concepts of FMP games and admissible trees.

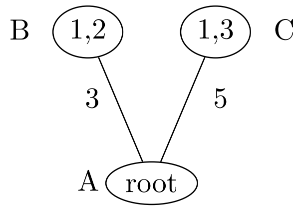

Figure 5.4: An FMP problem.

Example 5.2.2 In Figure 5.4 an FMP problem is depicted. The set of admissible trees for player 1 is $A_{\{1\}} = \{\{\{A,B\}\},\{\{A,C\}\},\{\{A,B\},\{A,C\}\}\}$. So $c(\{1\}) = \min\{3,5,3+5\} = 3$. Also note that $c(\{2,3\}) = 3+5 = 8$. $\diamond$

From the definition of FMP games it easily follows that FMP games are monotone games since for every $S \subseteq T \subseteq N$, we have $A_S \supseteq A_T$.

Note that in the tree of Figure 5.2, the position of player 1 in the vertex also containing player 2 seems irrelevant since the path from the root to this vertex contains another vertex occupied by player 1. We formalise this idea below. Given an FMP problem $\Gamma = (N,(V,E),0,S,a)$ we define the *reduced problem* $\Gamma^{red} = (N,(V,E),0,S^{red},a)$, where for every $v \in V$

$$S^{red}(v) = \{i \in S(v) : \text{ there is no } v' \in P(v), v' \neq v \text{ with } i \in S(v')\}.$$

That is, from the set of players occupying vertex v, those which also occupy a preceding vertex are dropped. Observe that the single-located players remain in their initial vertex. The cost game associated with the reduced problem will be denoted by (N,c^{red}). This reduction is illustrated in Example 5.2.3.

Example 5.2.3 In Figure 5.5 the reduced FMP problem arising from the problem in Figure 5.2 is depicted. In the tree of Figure 5.2 player 2 can

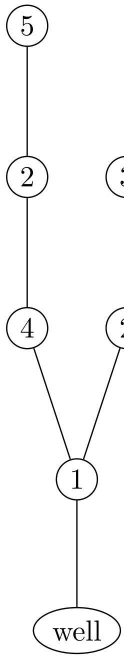

Figure 5.5: The reduced problem arising from the problem in Figure 5.2.

choose among four different paths to be connected to the root. However, the path that ends in the vertex occupied by players 1 and 2 is part of the path that ends in the vertex occupied by players 2 and 3. Therefore player 2 will never choose this second path to connect himself to the root, since this path yields a higher cost. So we can delete player 2 from the vertex occupied by players 2 and 3 without changing the game. Proceeding in this way we obtain the reduced problem which is depicted in Figure 5.5. Note that this problem is not a standard FMP problem since it still contains a player located in two vertices. $\diamond$

The proof of the following proposition is straightforward and therefore omitted.

Proposition 5.2.1 Let $(N,(V,E),0,S,a)$ be an FMP problem, and $(N,(V,E),0,S^{red},a)$ be its corresponding reduced problem. Let (N,c) and (N,c^{red}) be the associated cost games. Then (N,c) and (N,c^{red}) coincide.

Henceforth we assume in the remainder of this chapter, without loss of generality, that FMP problems are reduced.

5.3 Core and concavity

In this section we show that the core of an FMP game coincides with the core of a related standard FMP game. Since the core of each standard FMP game is non-empty we then conclude that FMP games have non-empty cores. Furthermore we investigate which marginal vectors are core elements and conclude that only standard FMP games are concave.

First we show that FMP games have non-empty cores. Consider the FMP problem $\Gamma = (N, (V, E), 0, S, a)$. We obtain the related standard FMP problem Γ^{st} by relocating the multi-located players. In particular, each multi-located player gets relocated to precisely one vertex. This new situation is defined by the 5-tuple $\Gamma^{st} = (N, (V, E), 0, S^{st}, a)$, where $S^{st}(v)$ is obtained from $S(v)$ as follows:

1. If player $i \in N$ is a multi-located player in Γ, then there is more than one path connecting him to the root. Since (V, E) is a tree, the common part of all these paths is again a path. Let $v^* \in V$ be the furthest vertex from the root on this common path. Then, in Γ^{st}, player i is located only in vertex v^*;

2. If player $i \in N$ is single-located in Γ, then in Γ^{st} this player remains in the same vertex.

In the following example the construction of Γ^{st} is illustrated.

Example 5.3.1 Consider the FMP problem Γ depicted in Figure 5.5. We obtain the related standard FMP problem Γ^{st}, which is depicted in Figure 5.6, by changing the position of player 2. Player 2 can choose between two paths in order to be connected to the root. The intersection of these two paths is the path which contains the root and the vertex occupied by player 1. In Γ^{st} player 2 is located in the furthest vertex from the root on this intersection path, i.e, the vertex occupied by player 1. $\diamond$

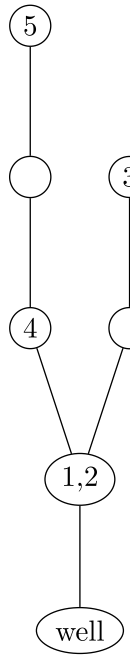

Figure 5.6: The standard FMP problem arising from the FMP problem of Figure 5.5.

In Granot, Maschler, Owen, and Zhu (1996) it is shown that fixed tree games are concave. The following proposition weakly generalises this result and can be proved in a similar way. Therefore the proof is omitted.

Proposition 5.3.1 Let $\Gamma = (N, (V, E), 0, S, a)$ be a standard FMP problem and (N, c) its corresponding game. Then (N, c) is concave.

In the following theorem we show that the cores of the cost games associated with an FMP problem and its corresponding standard FMP problem coincide.

Theorem 5.3.1 Let $\Gamma = (N, (V, E), 0, S, a)$ be an FMP problem and let $\Gamma^{st} = (N, (V, E), 0, S^{st}, a)$ be the corresponding standard FMP problem. Let (N, c) and (N, c^{st}) be the associated cost games. Then, $C(c) = C(c^{st})$.

Proof: First note that $c^{st}(S) \le c(S)$ for every $S \subseteq N$ and $c^{st}(N) = c(N)$. Consequently, $C(c^{st}) \subseteq C(c)$.

Now we show that $C(c) \subseteq C(c^{st})$. Let $x \in C(c)$. From the monotony of (N, c) it follows that $x \ge 0$. We need to show that for every $S \subseteq N$, $\sum_{i\in S} x_i \le c^{st}(S)$, or equivalently that $\sum_{i\in N\setminus S} x_i \ge c(N) - c^{st}(S)$.

Let $S \subseteq N$, and let V_S be the set of vertices in which the members of S are located in Γ^{st}, i.e. $V_S = \{v \in V | S^{st}(v) \cap S \neq \emptyset\}$. Let T_S be the smallest trunk containing V_S, and let E_S be the subset of edges corresponding to this trunk. By definition of T_S, $c^{st}(S) = \sum_{e\in E_S} a(e)$. Now let O_S denote the set of outgoing edges of T_S, i.e. $O_S = \{\{i, j\} \in E | i \in T_S \text{ and } j \notin T_S\}$. Furthermore let, for all $e \in O_S$, V_e be the set of vertices corresponding to edges of $F(e)\setminus\{e\}$. Finally, let $I_e^c = \bigcup_{v\notin V_e} S(v)$ and let $I_e = N\setminus I_e^c$. In other words, I_e are those players which appear only in vertices of V_e, and I_e^c is its complement. Because of Assumption (A2) it follows that $I_e \neq \emptyset$ for each $e \in O_S$.

Let $e \in O_S$. Since each member of I_e^c appears at least once in a vertex of $V\setminus V_e$, the edges in $F(e)$ are not needed to connect the members of I_e^c to the root. Hence, $c(I_e^c) \le \sum_{f\in E, f\notin F(e)} a(f)$ and therefore $\sum_{i\in I_e^c} x_i \le \sum_{f\in E, f\notin F(e)} a(f)$. From the efficiency of x we deduce that

$$\sum_{i\in I_e} x_i \ge c(N) - \sum_{f\in E, f\notin F(e)} a(f) = \sum_{f\in F(e)} a(f). \tag{5.1}$$

Hence, the players which appear only in one branch of the tree pay the entire cost of that branch. If $j \in S$, then it follows that $j \notin I_e$ for every $e \in O_S$. Therefore, $\bigcup_{e\in O_S} I_e \subseteq N\setminus S$. By definition of I_e it follows that $I_e \cap I_{\bar{e}} = \emptyset$ for all $e, \bar{e} \in O_S$, $e \neq \bar{e}$. That is, the I_e's are pairwise disjoint. Hence,

$$\sum_{e\in O_S} \sum_{f\in F(e)} a(f) \le \sum_{e\in O_S} \sum_{i\in I_e} x_i = \sum_{i\in \bigcup_{e\in O_S} I_e} x_i \le \sum_{i\in N\setminus S} x_i,$$

where the first inequality follows from (5.1), and the second from $x \ge 0$. From $\sum_{e\in O_S} \sum_{f\in F(e)} a(f) = \sum_{f\in E} a(f) - \sum_{f\in E_S} a(f) = c(N) - c^{st}(S)$, we now conclude that $\sum_{i\in N\setminus S} x_i \ge c(N) - c^{st}(S)$. □

The following corollary is an immediate consequence of Theorem 5.3.1 and the fact that concave games have non-empty cores.

Corollary 5.3.1 Let $\Gamma = (N, (V, E), 0, S, a)$ be an FMP problem and (N, c) its corresponding game. Then $C(c) \neq \emptyset$.

The next part of this section is dedicated to marginal vectors. We will characterise those orders whose corresponding marginal vectors are core elements. First we need some definitions. Let $\Gamma = (N, (V, E), 0, S, a)$ be an FMP problem and $\Gamma^{st} = (N, (V, E), 0, S^{st}, a)$ be the corresponding standard FMP problem. For every $i \in N$ we define

$$N_i = \{j \in N | \text{ there exist } v, v' \in V \text{ with } v' \in F(v),\, i \in S(v),\, j \in S^{st}(v')\}.$$

Note that for every single-located player $i \in N$ we have $i \in N_i$ and thus $N_i \neq \emptyset$. A coalition $S \subseteq N$ is called *proper* if for every $i \in S$ there exists a player $j \in S \cap N_i$. These definitions are illustrated in Example 5.3.2.

Example 5.3.2 Consider the FMP problem depicted in Figure 5.5. Player 2 is a multi-located player who appears in two different vertices, say v_1 and v_2. Consider these two vertices v_1 and v_2 in the standard problem and the corresponding sets of vertices $F(v_1)$ and $F(v_2)$. The set of players located in these sets of vertices in the standard problem is N_2. More precisely, $N_2 = \{3\} \cup \{5\} = \{3, 5\}$.

Consider coalition $S = \{2\}$. We can easily see that $S \cap N_2 = \emptyset$. Therefore coalition S is not proper. On the contrary, coalition $T = \{2, 3\}$ is proper. Since, 3 is single-located, $3 \in T \cap N_3$. It is also obvious that $3 \in T \cap N_2$. $\diamond$

The following lemma states that proper coalitions have the same cost in the FMP game and in its associated standard FMP game, while non-proper coalitions have a strictly larger cost in the FMP game than in its associated standard FMP game.

Lemma 5.3.1 Let $\Gamma = (N, (V, E), 0, S, a)$ be an FMP problem and let $\Gamma^{st} = (N, (V, E), 0, S^{st}, a)$ be the corresponding standard FMP problem. Let (N, c) and (N, c^{st}) be the associated cost games. Then, $c(S) = c^{st}(S)$ for all proper $S \subseteq N$ and $c(S) > c^{st}(S)$ for all non-proper $S \subseteq N$.

Proof: Let $S \subseteq N$. Let T^* be the optimal tree for S in Γ^{st} and let $V(T^*)$ be the vertex set corresponding to this tree. Since $T^* \subseteq T$ for every $T \in A_S(\Gamma)$, it is satisfied that $c(S) \geq c^{st}(S)$. Moreover, $c(S) = c^{st}(S)$ if and only if $T^* \in A_S(\Gamma)$. Hence, we need to prove that S is proper if and only if $T^* \in A_S(\Gamma)$.

First assume that S is proper. Let $i \in S$ and let $j \in N_i \cap S$. There are $v, v' \in V$ with $v' \in F(v)$, $i \in S(v)$ and $j \in S^{st}(v')$. Since T^* is optimal for S in Γ^{st} and $j \in S^{st}(v')$, it is satisfied that $v' \in V(T^*)$. Since $v' \in F(v)$ this implies that $v \in V(T^*)$. We conclude that T^* connects all $i \in S$ to the root in Γ. Therefore we have that $T^* \in A_S(\Gamma)$.

To show the reverse, assume that $T^* \in A_S(\Gamma)$. Let $i \in S$. Since T^* is admissible for S in Γ there is a $v \in V(T^*)$ with $i \in S(v)$. Because T^* is optimal for S in Γ^{st}, there is a $j \in S$ and a $v' \in V(T^*)$ with $v' \in F(v)$ and $j \in S^{st}(v')$. Thus $j \in S \cap N_i$. Hence, S is proper. $\square$

The next theorem characterises those orders whose corresponding marginal vectors are core elements.

Theorem 5.3.2 Let $(N, (V, E), 0, S, a)$ be an FMP problem, (N, c) the associated FMP game and $\sigma \in \Pi(N)$. Then, $m^\sigma(c) \in C(c)$ if and only if for every $i \in N$ there exists a $j \in N_i$ with $\sigma^{-1}(j) \leq \sigma^{-1}(i)$.

Proof: Let $(N, (V, E), 0, S^{st}, a)$ be the associated standard FMP problem, and (N, c^{st}) its associated game. First we show the "if" part. Assume that $\sigma \in \Pi(N)$ is such that for every player $i \in N$ there is a player $j \in N_i$ with $\sigma^{-1}(j) \leq \sigma^{-1}(i)$. Now let $k \in N$. Then, for all $i \in N$ with $\sigma^{-1}(i) \leq \sigma^{-1}(k)$ there is a $j \in N_i$ with $\sigma^{-1}(j) \leq \sigma^{-1}(i)$. Hence, $\sigma^{-1}(j) \leq \sigma^{-1}(k)$ and we conclude that $[\sigma(k), \sigma]$ is proper. This implies for all $S \subseteq N$,

$$\begin{aligned}
\sum_{k \in S} m_k^\sigma(c) &= \sum_{k \in S} \Big(c([\sigma(k), \sigma]) - c([\sigma(k-1), \sigma]) \Big) \\
&= \sum_{k \in S} \Big(c^{st}([\sigma(k), \sigma]) - c^{st}([\sigma(k-1), \sigma]) \Big) \\
&= \sum_{k \in S} m_k^\sigma(c^{st})
\end{aligned}$$

$$\begin{aligned} &\leq c^{st}(S) \\ &\leq c(S). \end{aligned}$$

The second equality follows by Lemma 5.3.1 and the fact that $[\sigma(k), \sigma]$ and $[\sigma(k-1), \sigma]$ are proper. The first inequality is due to concavity of (N, c^{st}) and the last inequality again by Lemma 5.3.1. We conclude that $m^\sigma(c) \in C(c)$.

Secondly, we show the "only if" part. Let $\sigma \in \Pi(N)$ be such that there is an $i \in N$ with $\sigma^{-1}(j) > \sigma^{-1}(i)$ for all $j \in N_i$. Consider coalition $S = [i, \sigma]$. Note that S is non-proper, because $S \cap N_i = \emptyset$. Then, by Lemma 5.3.1, $c(S) > c^{st}(S)$. Since $\sum_{i \in S} m_i^\sigma(c) = c(S)$, it follows that $\sum_{i \in S} m_i^\sigma(c) > c^{st}(S)$, and therefore $m^\sigma(c) \notin C(c^{st})$. Hence, by Theorem 5.3.1 we have $m^\sigma(c) \notin C(c)$. □

Let $\Gamma = (N, (V, E), 0, S, a)$ be a non-standard FMP problem and (N, c) its associated game. Let $i \in N$ be a multi-located player. Since i is multi-located, it follows that $i \notin N_i$. Now let $\sigma \in \Pi(N)$ be such that $\sigma(1) = i$. According to Theorem 5.3.2 it follows that $m^\sigma(c) \notin C(c)$ and we conclude that (N, c) is not concave. Hence, we obtain the following corollary.

Corollary 5.3.2 An FMP problem is standard if and only if the associated game is concave.

5.4 One-point solution concepts

In this section we consider three one-point solution concepts. First we remark that the nucleolus of an FMP game coincides with the nucleolus of its associated standard FMP game. Secondly, we consider the Shapley value and we show that for each non-standard FMP problem there exists a cost function on the edges such that the Shapley value is not a core element of the associated game. Furthermore we study the average of the extreme points of the core. We obtain a weight vector w such that if we divide the cost of each edge among its users proportionally to w, then we obtain an allocation that coincides with the average of extreme points of the core.

The nucleolus is a well known one-point solution concept introduced in Schmeidler (1969). The nucleolus has the property that it is a core element whenever the core is non-empty. In Potters and Tijs (1994) it is proved that if two games have the same core, with one of the games being concave, then both games have the same nucleolus as well. Hence, we conclude using Proposition 5.3.1 and Theorem 5.3.1 that the nucleolus of an FMP game coincides with the nucleolus of its associated standard FMP game.

Corollary 5.4.1 Let $\Gamma = (N, (V, E), 0, S, a)$ be an FMP problem and let Γ^{st} be its corresponding standard problem. Let (N, c) and (N, c^{st}) be the associated games. Then $nucleolus(c) = nucleolus(c^{st})$.

In the upcoming part of this section we study the Shapley value. We show that if an FMP problem contains a multi-located player, then there exists a cost function on the edges such that the Shapley value is not a core element of the associated game. In the proof we denote the set edges going out of $v \in V$ by O_v, i.e. $O_v = \{\{v, w\} \in E : w \in F(v)\}$.

Theorem 5.4.1 Let $(N, (V, E), 0, S)$ satisfy all relevant conditions of the definition of FMP problems. If $(N, (V, E), 0, S)$ contains a multi-located player, then there is an $a : E \to \mathbb{R}_{++}$ such that $\Phi(c) \notin C(c)$, with (N, c) the FMP game associated with $(N, (V, E), 0, S, a)$.

Proof: Let $i \in N$ be a multi-located player. Then there is more than one path connecting i to the root. Since (V, E) is a tree, the common part of all these paths is again a path. Let $v^* \in V$ be the furthest vertex from the root on this common path. Finally, let $p \geq 0$ be the number of edges on the path from the root to v^* and let $m \geq 1$ be the number of edges on the shortest path from v^* to a vertex where player i is located.

First suppose that $p = 0$. Let $a(e) = 1$ for all $e \in E$. Let $\Gamma = (N, (V, E), 0, S, a)$ be the corresponding FMP problem and (N, c) its associated FMP game. Furthermore, let $\Gamma^{st} = (N, (V, E), 0, S^{st}, a)$ be the corresponding standard FMP problem, and (N, c^{st}) its associated game. Since $p = 0$, $i \in S^{st}(0)$. That is, i is located at the root in Γ^{st}. Hence, we have that $c^{st}(\{i\}) = 0$. However, since at Γ player i is not located at the root, it follows that $\Phi_i(c) > 0$. Indeed, each $\sigma \in \Pi(N)$ with $\sigma(1) = i$ yields a marginal vector $m^\sigma(c)$ with $m_i^\sigma(c) > 0$. Furthermore, monotony of (N, c^{st}) implies that each marginal vector is non-negative. Hence, $\Phi_i(c) > 0 = c^{st}(\{i\})$, so $\Phi(c) \notin C(c^{st}) = C(c)$.

Now suppose that $p > 0$. Define

$$a(e) = \begin{cases} 1, & \text{if } e \in O_{v^*}; \\ \frac{1}{|N|p}, & \text{otherwise.} \end{cases}$$

Now consider the FMP problem $\Gamma = (N, (V, E), 0, S, a)$ and its associated game (N, c). Furthermore, let $\Gamma^{st} = (N, (V, E), 0, S^{st}, a)$ be the corresponding standard FMP problem and (N, c^{st}) its associated game. Now observe that $m_i^\sigma(c) = 1 + (m+p-1)\frac{1}{|N|p}$ for each $\sigma \in \Pi(N)$ with $\sigma(1) = i$. Note that there are $(|N|-1)!$ orders with $\sigma(1) = i$. Again we remark that because of monotony each marginal vector is non-negative. This yields

$$\Phi_i(c) \geq \frac{(|N|-1)!}{|N|!}(1 + (m+p-1)\frac{1}{|N|p}) > \frac{1}{|N|} = p\frac{1}{|N|p} = c^{st}(\{i\}).$$

We conclude that $\Phi(c) \notin C(c^{st}) = C(c)$. □

According to Theorem 5.4.1 the only FMP problems for which the Shapley value is a core element of the associated game, regardless of the cost function on the edges, are FMP problems without multi-located players. Therefore we consider an alternative solution concept, namely the average of the extreme points of the core. This average, denoted by $\alpha(c)$, is obviously a core element for each FMP game. Note that for each FMP game $\alpha(c)$ can be calculated straightforwardly, since each extreme point of $C(c)$ coincides with a marginal vector of a related standard FMP game. However, we provide an alternative method for obtaining $\alpha(c)$. In fact, we will introduce a weight vector w such that dividing the cost of each edge among its users proportionally to w yields the same allocation as $\alpha(c)$. First we develop some notation.

Let $\Gamma = (N, (V, E), 0, S, a)$ be a standard FMP problem and (N, c) its associated game. For each $e \in E$, let $v_e \in V$ be the endpoint of e located furthest away from the root. Furthermore, denote the set of users of edge $e \in E$ by I_e. For each $i \in N$, let $E_i \subseteq E$ be the set of edges on the path from i to the root. Define $n(e) = |S(v_e)| + |O_{v_e}|$ for each $e \in E$, and $w_i = \prod_{e \in E_i} \frac{1}{n(e)}$ for each $i \in N$. Finally, for each $e \in E$ and $i \in I_e$, let $w_i^e = \prod_{f \in E_i \cap F(e)} \frac{1}{n(f)}$. The following example illustrates these definitions.

Example 5.4.1 Let Γ be the standard FMP problem depicted in Figure 5.7. The edges are denoted by $f_1, \ldots, f_6$. Observe that, for instance, $n(f_1) = 4$, $n(f_2) = 2$ and $n(f_4) = 1$. Furthermore, $w_1 = w_2 = \frac{1}{4}$, $w_3 = \frac{1}{4}$, $w_4 = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$ and $w_5 = \frac{1}{8}$. Also observe that $w_4^{f_2} = \frac{1}{2}$. $\diamond$

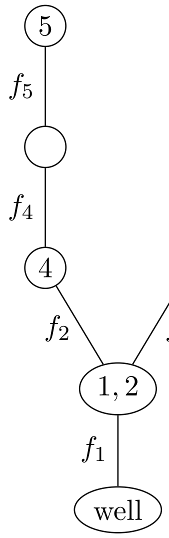

Figure 5.7: A standard FMP problem.

The remainder of this section is dedicated to showing that $\alpha_i(c) = \sum_{e \in E_i} \frac{w_i}{\sum_{j \in I_e} w_j} a(e)$ for each $i \in N$. In order to do so, we need two lemmas.

Lemma 5.4.1 For all $e \in E$, $\sum_{i \in I_e} w_i^e = 1$.

Proof: For each $e \in E$, let $k(e)$ denote the maximal length of a path from v_e to a leaf of (V, E). We prove the lemma by induction on $k(e)$.

Let $e \in E$ be such that $k(e) = 0$. Then v_e is a leaf of (V, E). Therefore, $|O_{v_e}| = 0$, $n(e) = |S(v_e)|$, $I_e = S(v_e)$ and $w_i^e = \frac{1}{n(e)}$ for each $i \in I_e$. This implies

$$\sum_{i \in I_e} w_i^e = \sum_{i \in S(v_e)} \frac{1}{n(e)} = |S(v_e)| \frac{1}{|S(v_e)|} = 1.$$

Now assume, as the induction hypothesis, that $\kappa \geq 1$ is such that $\sum_{i \in I_f} w_i^f = 1$ for each $f \in E$ with $k(f) < \kappa$. Let $e \in E$ be such that $k(e) = \kappa$. Note that for each $f \in O_{v_e}$ we have that $k(f) \le \kappa - 1 < \kappa$. This implies

$$\begin{aligned}
\sum_{i \in I_e} w_i^e &= \sum_{i \in S(v_e)} w_i^e + \sum_{f \in O_{v_e}} \sum_{i \in I_f} w_i^e \\
&= \sum_{i \in S(v_e)} \frac{1}{n(e)} + \sum_{f \in O_{v_e}} \sum_{i \in I_f} \frac{1}{n(e)} w_i^f \\
&= |S(v_e)| \frac{1}{n(e)} + \sum_{f \in O_{v_e}} \frac{1}{n(e)} \\
&= |S(v_e)| \frac{1}{n(e)} + |O_{v_e}| \frac{1}{n(e)} \\
&= 1.
\end{aligned}$$

The second equality is satisfied because $w_i^e = \frac{1}{n(e)}$ for each $i \in S(v_e)$, and because $w_i^e = \prod_{g \in E_i \cap F(e)} \frac{1}{n(g)} = \frac{1}{n(e)} \prod_{g \in E_i \cap F(f)} \frac{1}{n(g)} = \frac{1}{n(e)} w_i^f$ for each $f \in O_{v_e}$ and $i \in I_f$. The third equality is due to our induction hypothesis that $\sum_{i \in I_f} w_i^f = 1$ for each $f \in E$ with $k(f) < \kappa$. □

Lemma 5.4.2 For all $e \in E$ and $i \in I_e$, $w_i^e = \frac{w_i}{\sum_{j \in I_e} w_j}$.

Proof: Observe that

$$\begin{aligned}
w_i^e &= \frac{w_i^e}{\sum_{j \in I_e} w_j^e} \\
&= \frac{\prod_{f \in E_i \cap F(e)} \frac{1}{n(f)}}{\sum_{j \in I_e} \prod_{f \in E_j \cap F(e)} \frac{1}{n(f)}} \\
&= \frac{\prod_{f \in E_i, f \notin F(e)} \frac{1}{n(f)}}{\prod_{f \in E_i, f \notin F(e)} \frac{1}{n(f)}} \frac{\prod_{f \in E_i \cap F(e)} \frac{1}{n(f)}}{\sum_{j \in I_e} \prod_{f \in E_j \cap F(e)} \frac{1}{n(f)}} \\
&= \frac{\prod_{f \in E_i} \frac{1}{n(f)}}{\sum_{j \in I_e} \prod_{f \in E_i, f \notin F(e)} \frac{1}{n(f)} \prod_{f \in E_j \cap F(e)} \frac{1}{n(f)}} \\
&= \frac{\prod_{f \in E_i} \frac{1}{n(f)}}{\sum_{j \in I_e} \prod_{f \in E_j} \frac{1}{n(f)}} \\
&= \frac{w_i}{\sum_{j \in I_e} w_j}.
\end{aligned}$$

The first equality is satisfied because of Lemma 5.4.1. For the fifth equality we have used that $\{f \in E_j : f \notin F(e)\} = \{f \in E_i : f \notin F(e)\}$ for each $j \in I_e$. The last equality is satisfied by definition of w. □

It will be convenient to describe the set of extreme points of $C(c)$ in terms of so-called consistent edge-assignments. An *edge-assignment* is a map $\tau : E \rightarrow N$ that assigns each edge to precisely one player. An edge-assignment τ is *consistent* if $\tau(e) \in I_e$ for each $e \in E$, and if $\tau(e) = i$ and $f \in E_i \cap F(e)$ imply $\tau(f) = i$. In other words, $e \in E$ is assigned to a user of e, and if $e \in E$ is assigned to $i \in I_e$, then all edges on the path from e to i are assigned to i as well. Let Y be the set of consistent edge-assignments. With each $\tau \in Y$ we associate an allocation vector in the following straightforward way: $x_i^\tau = \sum_{e\in E:\tau(e)=i} a(e)$ for each $i \in N$. That is, each player pays for the edges that are assigned to him. Let $X^Y = \{x^\tau : \tau \in Y\}$. Since $a(e) > 0$ for each $e \in E$, it follows that the correspondence between Y and X^Y is a one-to-one correspondence. Hence, $|Y| = |X^Y|$.

We will now argue that the set of extreme points of $C(c)$ coincides with X^Y. Let X be the set of extreme points of $C(c)$ and let $x \in X$. Since x corresponds to a marginal vector, the cost of each edge is allocated to a single user. In fact, if the cost of edge e is allocated to player i, then the cost of all edges on the path from e to i are allocated to i. Hence, x can be associated with a consistent edge-assignment. We conclude that $X \subseteq X^Y$.

Now let $\tau \in Y$. Let $\sigma \in \Pi(N)$ be such that for all $i, j \in N$, $\sigma^{-1}(i) \leq \sigma^{-1}(j)$ if $\tau(e) = i$ for some $e \in E_j$. In other words, if an edge at the path from j to the root is assigned to i, then i is ordered before j. Observe, since τ is consistent, that σ exists. Indeed, for all $i, j \in N$, $i \neq j$, it cannot happen that we require that $\sigma^{-1}(i) \leq \sigma^{-1}(j)$ and $\sigma^{-1}(j) \leq \sigma^{-1}(i)$. Finally, note that $m^\sigma(c) = x^\tau$. We conclude that $X^\tau \subseteq X$.

The coincidence between X and X^Y implies that $\alpha(c) = \frac{1}{|X|}\sum_{x\in X} x = \frac{1}{|Y|}\sum_{\tau\in Y} x^\tau$. We will now show that $\alpha_i(c) = \sum_{e\in E_i} \frac{w_i}{\sum_{j\in I_e} w_j} a(e)$.

Theorem 5.4.2 Let $\Gamma = (N, (V, E), 0, S, a)$ be a standard fixed tree problem and (N, c) be the associated fixed tree game. Then $\alpha_i(c) = \sum_{e\in E_i} \frac{w_i}{\sum_{j\in I_e} w_j} a(e)$ for each $i \in N$.

Proof: First observe that

$$\alpha_i(c) = \frac{1}{|Y|}\sum_{\tau\in Y} x_i^\tau = \frac{1}{|Y|}\sum_{\tau\in Y}\sum_{e\in E:\tau(e)=i} a(e) = \frac{1}{|Y|}\sum_{e\in E_i}\sum_{\tau\in Y:\tau(e)=i} a(e).$$

We claim that $|Y| = \prod_{e \in E} n(e)$. Indeed, at a consistent edge-assignment edge $e \in E$ can be assigned to precisely $n(e)$ players. In particular, it can be assigned to a player in $|S(v_e)|$, or to player $\tau(f) \in I_f$, $f \in O_{v_e}$. By definition of consistency it is not allowed to assign the cost of $e \in E$ to $j \in I_f \backslash \{\tau(f)\}$.

Similarly it can be seen that $|\{\tau \in Y : \tau(e) = i\}| = \prod_{f \in E : f \notin E_i \cap F(e)} n(f)$ for each $e \in E$ and $i \in I_e$. If $e \in E$ is assigned to i, then, because of consistency, all edges in $E_i \cap F(e)$ are assigned to player i as well. The remaining edges can be assigned in $\prod_{f \in E : f \notin E_i \cap F(e)} n(f)$ different ways.

We conclude that

$$\begin{aligned}
\alpha_i(c) &= \frac{1}{|Y|} \sum_{e \in E_i} \sum_{\tau \in Y : \tau(e) = i} a(e) \\
&= \sum_{e \in E_i} a(e) \frac{\prod_{f \in E : f \notin E_i \cap F(e)} n(f)}{\prod_{f \in E} n(f)} \\
&= \sum_{e \in E_i} a(e) \frac{1}{\prod_{f \in E_i \cap F(e)} n(f)} \\
&= \sum_{e \in E_i} a(e) w_i^e \\
&= \sum_{e \in E_i} \frac{w_i}{\sum_{j \in I_e} w_j} a(e).
\end{aligned}$$

The last equality is satisfied because of Lemma 5.4.2. □

Example 5.4.2 Let Γ be the FMP problem depicted in Figure 5.7 with $a : E \to \mathbb{R}_{++}$ the cost function on the edges. Let (N, c) be the associated FMP game. Then, $\alpha_1(c) = \frac{1}{4}a(f_1)$, $\alpha_2(c) = \frac{1}{4}a(f_1)$, $\alpha_3(c) = \frac{1}{4}a(f_1) + a(f_3) + a(f_6)$, $\alpha_4(c) = \frac{1}{8}a(f_1) + \frac{1}{2}a(f_2)$, and $\alpha_5(c) = \frac{1}{8}a(f_1) + \frac{1}{2}a(f_2) + a(f_4) + a(f_5)$. ◇

Chapter 6

Sequencing games

6.1 Introduction

In operations research, sequencing situations are characterised by a finite number of jobs, lined up in front of one (or more) machine(s), that have to be processed on the machine(s). A single decision maker wants to determine a processing order of the jobs that minimises a cost criterion and takes into account possible restrictions on the jobs (e.g. due dates, precedence constraints, etc.) This single decision maker problem can be transformed into a multiple decision maker problem by taking agents into account who own at least one job. In such a model a group of agents (coalition) can save costs by cooperation. The question then arises how to divide the total cost savings among the group of agents. This question was first addressed in Curiel, Pederzoli, and Tijs (1989). In that paper sequencing games are introduced, arising from sequencing situations where weighted completion times are the cost criteria of the agents. It was shown that sequencing games are convex, and thus have non-empty cores. Moreover, an allocation rule dividing the total cost savings obtained from complete cooperation was introduced and characterised.

The paper by Curiel, Pederzoli, and Tijs (1989) turned out to be the starting point of a vast growing literature on the interaction between scheduling theory and cooperative game theory. Van den Nouweland, Krabbenborg, and Potters (1992), Hamers, Klijn, and Suijs (1999) and Calleja, Borm, Hamers, Klijn, and Slikker (2002) investigate sequencing games that arise from multiple-machine sequencing situations. These papers mainly focus on the non-emptiness of the core of the related sequencing games.

Hamers, Borm, and Tijs (1995) introduce sequencing games arising from situations with ready times (release dates) on the jobs. In this case the corresponding sequencing games have non-empty cores, but are not necessarily convex. For a special subclass, however, convexity could be established. Similar results are obtained in Borm, Fiestras-Janeiro, Hamers, Sánchez, and Voorneveld (2002), in which due dates are imposed on the jobs.

The remainder of this chapter is organised as follows. In Section 6.2 we introduce sequencing situations and games as introduced in Curiel, Pederzoli, and Tijs (1989).

In Section 6.3, which is based on Van Velzen (2004b), sequencing games are introduced arising from situations with controllable processing times. In particular, it is assumed that jobs can be processed in shorter durations, but at extra cost. The main focus is non-emptiness of the core. Convexity is studied for some special instances.

Section 6.4 is based on Hamers, Klijn, and Van Velzen (2005). That section introduces sequencing games arising from situations with a precedence relation on the jobs. Convexity is shown for games arising from situations where the precedence relation is a network of parallel chains, and the initial order is a concatenation of these chains.

Weak-relaxed sequencing games are introduced in Section 6.5. These sequencing games arise from sequencing situations by considering a less restricted set of admissible processing orders. The section, based on Van Velzen and Hamers (2003), shows non-emptiness of the core of these games.

Finally, in Section 6.6, which is based on Hamers, Klijn, Slikker, and Van Velzen (2004), we introduce a model where a finite number of indivisible objects need to be allocated to the same number of agents. We assume that the agents are allowed to choose the objects according to some prescribed order. Furthermore we assume that the agents will collaborate in order to achieve a society-efficient assignment of the objects and we show that this assignment is supported by side-payments that guarantee stability, i.e. such that each group of agents has an incentive to collaborate with the other agents.

6.2 Sequencing situations and games

In this section we introduce sequencing situations and games as introduced in Curiel, Pederzoli, and Tijs (1989).

In a *sequencing situation* there is a queue of agents, each with one job, in front of a machine. Each agent has to process his job on the machine. The set of agents is denoted by N. The positive *processing time* p_i of the job of agent $i \in N$ is the time the machine takes to handle this job. Each agent has a *cost function* that is linear in the completion time of its job. In particular, if t is the completion time of the job of agent i, then $c_i(t) = \alpha_i t$ with $\alpha_i > 0$. We assume that the machine can only handle one job at once, and that processing schedules are *semi-active*, i.e. there are no idle times in between the processing of jobs. Hence, a *processing schedule* is merely an order $\sigma : \{1, \ldots, |N|\} \to N$ that specifies the positions of the jobs in the queue. The set of all processing orders is denoted by $Pr(N)$. We assume there is an exogenously given *initial processing order* $\sigma_0 \in Pr(N)$. This initial processing order has the interpretation that the jobs will be processed according to this order, unless the agents decide to reorder their jobs. A sequencing situation is formally described by the tuple (N, σ_0, α, p).

If $\sigma \in Pr(N)$ is the processing order of the jobs, then $\sum_{j \in \{1,\ldots,|N|\}: j \leq \sigma^{-1}(i)} p_{\sigma(j)}$ is the *completion time* of job $i \in N$. Therefore, the cost of agent $i \in N$ at processing order σ is given by

$$C_i(\sigma) = \alpha_i \Bigg(\sum_{j \in \{1,\ldots,|N|\}: j \leq \sigma^{-1}(i)} p_{\sigma(j)} \Bigg).$$

A processing order is called *optimal* if it minimises the sum of the costs of all agents. Formally, $\sigma \in Pr(N)$ is optimal if

$$\sum_{i \in N} C_i(\sigma) \leq \sum_{i \in N} C_i(\tau) \text{ for each } \tau \in Pr(N).$$

In Smith (1956) it is shown that a processing order is optimal if and only if the jobs are lined up in decreasing order of urgency indices, where the *urgency index* of $i \in N$ is given by $u_i = \frac{\alpha_i}{p_i}$. We refer to this result as the *Smith-rule*.

In the remainder of this section we formally introduce sequencing games. The characteristic function of a sequencing game describes the maximal cost savings that coalitions can obtain by means of admissible rearrangements of the initial processing order. Obviously, the worth of a coalition depends on the interpretation of the clause "admissible rearrangements." We call a processing order $\sigma \in Pr(N)$ *admissible* for coalition $S \subseteq N$ if

$$\begin{aligned} &\{j \in N : \sigma^{-1}(j) \leq \sigma^{-1}(i)\} \\ =&\{j \in N : \sigma_0^{-1}(j) \leq \sigma_0^{-1}(i)\} \text{ for each } i \in N \backslash S. \end{aligned} \tag{6.1}$$

This condition implies that the completion times of the players outside the coalition remain the same. Furthermore, players in S are not allowed to jump over players outside S. The set of all admissible processing orders for coalition $S \subseteq N$ is denoted by $A(S)$. The *(standard) sequencing game* (N, v) is now defined by

$$v(S) = \sum_{i \in S} C_i(\sigma_0) - \min_{\sigma \in A(S)} \sum_{i \in S} C_i(\sigma)$$

for each $S \subseteq N$. It can be easily seen that sequencing games are superadditive. In fact, sequencing games are chain-component additive with respect to σ_0.[5] This result follows directly from admissibility condition (6.1).

We call $S \subseteq N$ connected with respect to σ_0 if there are $i, j \in \{1, \ldots, |N|\}$ with $S = \{\sigma_0(i), \ldots, \sigma_0(j)\}$. If a coalition is not connected, then it consists of several connected components. We denote the components of $S \subseteq N$ with respect to σ_0 by $\mathcal{C}(S)$.

Let $\sigma \in Pr(N)$ and let $i, j \in N$ be such that agent i is directly in front of agent j with respect to σ. The cost savings that can be obtained by switching i and j equal $g_{ij} = \max\{0, \alpha_j p_i - \alpha_i p_j\}$. Observe that this amount is positive if and only if $u_j > u_i$. Since optimal processing orders

[5]This is a small abuse of terminology since we introduced component additivity of games with respect to trees and not with respect to orders. However, the initial order σ_0 constitutes a chain in a straightforward way.

can be reached by series of switches of neighbours, it follows that

$$v(S) = \sum_{T \in \mathcal{C}(S)} \sum_{i,j \in T : i<j} g_{ij},$$

for each $S \subseteq N$. Using this last expression it is easily verified that sequencing games are convex, and therefore have non-empty cores (cf. Curiel, Pederzoli, and Tijs (1989)).

6.3 Sequencing games with controllable processing times

In reality processing jobs does not only require a machine, but also additional resources such as manpower, funds, etc. This implies that jobs can be processed in shorter or longer durations by increasing or decreasing these additional resources. Of course, deploying these additional resources entails extra costs, but these extra costs might be compensated by the gains from job completion at an earlier time. Sequencing situations with controllable processing times, or cps situations for short, are investigated in, among others, Vickson (1980a), Vickson (1980b) and Alidaee and Ahmadian (1993). An overview of literature on cps situations is given in Nowicki and Zdrzalka (1990).

In this section, which is based on Van Velzen (2004b), we consider sequencing games arising from cps situations. For these so-called cps games we obtain two core elements that depend only on an optimal schedule for the grand coalition. Furthermore we show that many marginal vectors are core elements, in spite of the fact that these games are not convex in general. Finally, we consider convexity for some special instances of cps games.

6.3.1 Sequencing situations with controllable processing times and games

A sequencing situation with controllable processing times, or cps situation for short, is a tuple $(N, \sigma_0, \alpha, \beta, p, \bar{p})$. Here, N, σ_0, α and p have the same interpretation as in Section 6.2. For the sake of notational simplicity we assume throughout this section that $\sigma_0(i) = i$ for each $i \in \{1, \ldots, |N|\}$. However, in this section we assume that the processing times of the jobs are not fixed. In particular, the processing time p_i of $i \in N$ can be reduced to at most $\bar{p}_i$, the *crashed processing time* of i. The amount of time by which the processing time of i is reduced is called the *crash time* of i. We assume that $0 \leq \bar{p}_i \leq p_i$ for each $i \in N$. The cost of each agent is linear in the completion time of its job, and in the crash time of its job. That is, if t is the completion time and y the crash time of job i, then

$$C_i(t, y) = \alpha_i t + \beta_i y.$$

Of course, α_i and β_i are both positive constants for each $i \in N$. Since crashing a job requires additional resources, we assume that $\alpha_i \leq \beta_i$ for all $i \in N$. That is, reducing the processing time of a job by one time unit costs more than the processing of that job by one time unit.

Since processing times are not fixed, a *processing schedule* is a pair (σ, x) with $\sigma \in Pr(N)$ and x a vector of feasible processing times. At processing schedule (σ, x), the completion time of job $i \in N$ is equal to $\sum_{j \in \{1,\ldots,|N|\}: j \leq \sigma^{-1}(i)} x_{\sigma(j)}$, and its crash time is $p_i - x_i$. Hence, the cost of agent $i \in N$ at processing schedule (σ, x) is

$$C_i(\sigma, x) = \alpha_i\Big(\sum_{j \in \{1,\ldots,|N|\}: j \leq \sigma^{-1}(i)} x_{\sigma(j)}\Big) + \beta_i(p_i - x_i).$$

A processing schedule (σ, x) is called *optimal* if it minimises the sum of the costs of all agents, i.e. if

$$\sum_{i \in N} C_i(\sigma, x) \leq \sum_{i \in N} C_i(\tau, y) \text{ for any processing schedule } (\tau, y).$$

Finding an optimal processing schedule for a cps situation falls into the class of NP-hard problems (Hoogeveen and Woeginger (2002)). The difficulty of this problem lies in finding optimal processing times. Once a vector of optimal processing times is known, it is straightforward to find a corresponding optimal processing order by applying the Smith-rule. Although finding optimal processing schedules is difficult, the following lemma, due to Vickson

(1980a), is helpful for our purposes. This lemma states that there is an optimal processing schedule such that the processing time of each job is either equal to its initial processing time, or its crashed processing time. We note that this result easily follows from the linearity of the cost functions of the agents.

Lemma 6.3.1 (Vickson (1980a)) Let $(N, \sigma_0, \alpha, \beta, p, \bar{p})$ be a cps situation. There exists an optimal processing schedule (σ, x) such that $x_i \in \{p_i, \bar{p}_i\}$ for all $i \in N$.

From Lemma 6.3.1 it follows that an optimal processing schedule can be found by considering all $2^{|N|}$ possibilities for the processing times and applying the Smith-rule for each of these possibilities. Without loss of generality we assume throughout this section that optimal processing schedules satisfy the property of Lemma 6.3.1, i.e. if (σ, x) is an optimal processing schedule, then $x_i \in \{p_i, \bar{p}_i\}$ for all $i \in N$.

In the remainder of this section we introduce sequencing games with controllable processing times, or cps games for short. Let $(N, \sigma_0, \alpha, \beta, p, \bar{p})$ be a cps situation. The characteristic function of a cps game will express the maximal cost savings each coalition can obtain by means of an admissible alteration of the initial processing schedule. For this we have to agree upon which processing schedules are admissible for a coalition. We call a processing schedule admissible for a coalition if it satisfies the following three properties. First, the processing times of the players belonging to the coalition should be feasible, i.e. in between the crashed processing time and the initial processing time. Secondly, the processing times of players outside the coalition should remain unchanged. Finally, the schedule should be such that the jobs outside the coalition remain in their initial position and no jumps take place over players outside the coalition. Let $AS(S)$ denote the set of admissible schedules for coalition $S \subseteq N$. Mathematically, $(\sigma, x) \in AS(S)$ if

$$\bar{p}_i \leq x_i \leq p_i \quad \text{for all} \quad i \in S \tag{6.2}$$

$$x_i = p_i \quad \text{for all} \quad i \in N \backslash S \tag{6.3}$$

and if σ satisfies (6.1). The *cps game* (N, v) is now defined by

$$v(S) = \sum_{i \in S} C_i(\sigma_0, p) - \min_{(\sigma,x) \in AS(S)} \sum_{i \in S} C_i(\sigma, x)$$

for each $S \subseteq N$. The following lemma shows that cps games are superadditive.

Lemma 6.3.2 Let $(N, \sigma_0, \alpha, \beta, p, \bar{p})$ be a cps situation and let (N, v) be the corresponding game. Then (N, v) is superadditive.

Proof: Let $S, T \subseteq N$ be non-empty sets with $S \cap T = \emptyset$. Let $(\sigma^S, x^S) \in AS(S)$ be an optimal processing schedule for coalition S, and $(\sigma^T, x^T) \in AS(T)$ be an optimal processing schedule for coalition T. Now let $x^{S \cup T}$ be given by

$$x_i^{S \cup T} = \begin{cases} x_i^S, & \text{if } i \in S; \\ x_i^T, & \text{if } i \in T; \\ p_i, & \text{if } i \in N \backslash (S \cup T); \end{cases}$$

Furthermore, let $\sigma^{S \cup T} \in Pr(N)$ be a "merger" between σ^S and σ^T, i.e. let $\sigma^{S \cup T} \in Pr(N)$ be such that

$$(\sigma^{S \cup T})^{-1}(i) = \begin{cases} (\sigma^S)^{-1}(i), & \text{if } i \in S; \\ (\sigma^T)^{-1}(i), & \text{if } i \in T; \\ \sigma_0^{-1}(i), & \text{if } i \in N \backslash (S \cup T); \end{cases}$$

Now observe that $(\sigma^{S \cup T}, x^{S \cup T}) \in AS(S \cup T)$. It is easily verified that $C_i(\sigma^{S \cup T}, x^{S \cup T}) \leq C_i(\sigma^S, x^S)$ for each $i \in S$. Indeed, the position of player $i \in S$ is the same at both orders, and the crash time of i is the same as well. However, the completion time of i at $(\sigma^{S \cup T}, x^{S \cup T})$ might be smaller than at (σ^S, x^S), since at $(\sigma^{S \cup T}, x^{S \cup T})$ player i might benefit from possible crashes of jobs corresponding to players in T. Similarly, $C_i(\sigma^{S \cup T}, x^{S \cup T}) \leq C_i(\sigma^T, x^T)$

for each $i \in T$. It is straightforward to verify that

$$\begin{aligned}
v(S \cup T) &= \sum_{i \in S \cup T} C_i(\sigma_0, p) - \min_{(\sigma,x) \in AS(S \cup T)} \sum_{i \in S \cup T} C_i(\sigma, x) \\
&\geq \sum_{i \in S \cup T} C_i(\sigma_0, p) - \sum_{i \in S \cup T} C_i(\sigma^{S \cup T}, x^{S \cup T}) \\
&= \sum_{i \in S} C_i(\sigma_0, p) - \sum_{i \in S} C_i(\sigma^{S \cup T}, x^{S \cup T}) \\
&\quad + \sum_{i \in T} C_i(\sigma_0, p) - \sum_{i \in T} C_i(\sigma^{S \cup T}, x^{S \cup T}) \\
&\geq \sum_{i \in S} C_i(\sigma_0, p) - \sum_{i \in S} C_i(\sigma^S, x^S) \\
&\quad + \sum_{i \in T} C_i(\sigma_0, p) - \sum_{i \in T} C_i(\sigma^T, x^T) \\
&= v(S) + v(T).
\end{aligned}$$

The first inequality is satisfied because $(\sigma^{S \cup T}, x^{S \cup T})$ need not be optimal for $S \cup T$. □

Similarly to Lemma 6.3.1 it is straightforward to see that for each coalition $S \subseteq N$ there is an optimal processing schedule with the processing time of each job either equal to its initial processing time, or to its crashed processing time. Therefore we assume throughout this section that optimal processing schedules satisfy this property. We now give an example of cps games. This example illustrates that cps games need not be chain-component additive with respect to σ_0 nor convex.

Example 6.3.1 Let $(N, \sigma_0, \alpha, \beta, p, \bar{p})$ be given by $N = \{1, 2, 3, 4\}$, $\alpha = (1, 1, 1, 1)$, $\beta = (2, 2, 2, 2)$, $p = (10, 4, 3, 15)$, $\bar{p} = (4, 3, 2, 5)$, and let (N, v) be the corresponding cps game. Now consider for instance coalition $\{1, 2, 3\}$. An optimal schedule for this coalition is given by $(\sigma, x) \in AS(\{1, 2, 3\})$ with $\sigma = (3, 2, 1, 4)$ and $x = (10, 4, 2, 15)$. This yields cost savings $v(\{1, 2, 3\}) = 15$.

It can be verified that $v(\{1\}) = v(\{1, 3\}) = v(\{3, 4\}) = 0$, $v(\{1, 3, 4\}) = 6$ and $v(N) = 17$. We conclude that $v(\{1\}) + v(\{3, 4\}) = 0 < 6 = v(\{1, 3, 4\})$.

Hence, (N, v) is not chain-component additive with respect to σ_0. Furthermore, $v(\{1,2,3\}) + v(\{1,3,4\}) = 21 > 17 = v(N) + v(\{1,3\})$. This shows that (N, v) is not convex. $\diamond$

Note that non-emptiness of the core of many extensions of standard sequencing games is proved by means of chain-component additivity with respect to σ_0 (e.g. Borm, Fiestras-Janeiro, Hamers, Sánchez, and Voorneveld (2002) and Hamers, Borm, and Tijs (1995)). Since cps games need not be chain-component additive with respect to σ_0 nor convex, another approach is needed to establish non-emptiness of the core. This will be the main issue of the upcoming section.

6.3.2 Cores of sequencing games with controllable processing times

In this section we prove non-emptiness of the core of cps games. In particular, we provide two core elements that depend only on an optimal processing schedule for the grand coalition. Furthermore we show that many marginal vectors are core elements.

In the first part of this section we provide two core elements that only depend on an optimal processing schedule for the grand coalition. For the perception of these two core elements it is important to note that the optimal processing schedule for the grand coalition can be reached from the initial processing schedule in several ways. For example one could first reduce the initial processing times of the jobs to the optimal processing times, and then rearrange the jobs. But one could also first rearrange the jobs and then reduce the initial processing times to the optimal processing times. We emphasise these two possibilities since the construction of our two core elements depends on them.

Let $(N, \sigma_0, \alpha, \beta, p, \bar{p})$ be a cps situation. Let $(\sigma, x) \in AS(N)$ be an optimal processing schedule for the grand coalition. For our first core element, denoted by $\gamma(\sigma, x)$, we will reach this optimal processing schedule by first crashing jobs, and then rearranging them. Let $\gamma(\sigma, x)$ be obtained as follows. First give the cost savings (or costs) obtained by the crashing of a job to the job that crashes and secondly, allocate the cost savings obtained by interchanging two jobs to the back job. Or to put it in a formula:

$$\gamma_i(\sigma, x) = \Big(\sum_{j \in N: j \geq i} \alpha_j - \beta_i\Big)(p_i - x_i) + \sum_{j \in N: j < i} (\alpha_i x_j - \alpha_j x_i)_+,$$

for all $i \in N$. Here $(\sum_{j \in N: j \geq i} \alpha_j - \beta_i)(p_i - x_i)$ are the cost savings obtained by crashing job i by $p_i - x_i$ units of time. The cost savings obtained by moving job i up in the queue equal $\sum_{j \in N: j < i}(\alpha_i x_j - \alpha_j x_i)_+$.

For our second core element, denoted by $\delta(\sigma, x)$, we reach the optimal processing schedule (σ, x) by first interchanging jobs to the optimal order, and then crashing them. Let $\delta(\sigma, x)$ be the following allocation. First give the cost savings (or costs) of each neighbourswitch to the back job, and secondly, allocate the cost savings due to the crashing of a job to the job that crashes. That is,

$$\begin{aligned} \delta_i(\sigma, x) \quad = & \sum_{j \in N: j < i, \sigma^{-1}(j) > \sigma^{-1}(i)} (\alpha_i p_j - \alpha_j p_i) \\ & + \Big(\sum_{k \in N: \sigma^{-1}(k) \geq \sigma^{-1}(i)} \alpha_k - \beta_i\Big)(p_i - x_i). \end{aligned}$$

for each $i \in N$. Clearly, $\gamma(\cdot)$ and $\delta(\cdot)$ depend on which optimal processing schedule is used. However, because of notational convenience we will write γ and δ instead of $\gamma(\sigma, x)$ and $\delta(\sigma, x)$ if there can be no confusion about the optimal processing schedule that is used.

Before we show that γ and δ are core elements, we need two lemmas. In the first we obtain an expression for δ that we use in the proof of Theorem 6.3.1. The second lemma is a technical but straightforward lemma which we use throughout this section.

Lemma 6.3.3 Let $(N, \sigma_0, \alpha, \beta, p, \bar{p})$ be a cps situation and let $(\sigma, x) \in AS(N)$ be an optimal processing schedule. Then,

$$\begin{aligned} \delta_i \quad = \quad & \gamma_i - \sum_{j \in N: j > i, \sigma^{-1}(j) < \sigma^{-1}(i)} \alpha_j (p_i - x_i) \\ & + \sum_{j \in N: j < i, \sigma^{-1}(j) > \sigma^{-1}(i)} \alpha_i (p_j - x_j). \end{aligned}$$

for all $i \in N$.

Proof: Let $i \in N$. Then

$$\begin{aligned}\delta_i \;=\; & \sum_{j \in N: j<i,\ \sigma^{-1}(j)>\sigma^{-1}(i)} (\alpha_i p_j - \alpha_j p_i) && (6.4)\\ & + \Bigg(\sum_{k \in N: \sigma^{-1}(k) \geq \sigma^{-1}(i)} \alpha_k - \beta_i \Bigg)(p_i - x_i) && (6.5)\end{aligned}$$

First note for expression (6.4) that

$$\begin{aligned}& \sum_{j \in N: j<i,\ \sigma^{-1}(j)>\sigma^{-1}(i)} (\alpha_i p_j - \alpha_j p_i)\\ = & \sum_{j \in N: j<i,\ \sigma^{-1}(j)>\sigma^{-1}(i)} (\alpha_i x_j - \alpha_j p_i) && (6.6)\\ & + \sum_{j \in N: j<i,\ \sigma^{-1}(j)>\sigma^{-1}(i)} \alpha_i (p_j - x_j). && (6.7)\end{aligned}$$

Furthermore, note for expression (6.5) that

$$\begin{aligned}& \Bigg(\sum_{k \in N: \sigma^{-1}(k) \geq \sigma^{-1}(i)} \alpha_k - \beta_i \Bigg)(p_i - x_i)\\ = & \Bigg(\sum_{k \in N: k \geq i} \alpha_k - \beta_i \Bigg)(p_i - x_i) && (6.8)\\ & - \sum_{j \in N: j>i,\ \sigma^{-1}(j)<\sigma^{-1}(i)} \alpha_j (p_i - x_i) && (6.9)\\ & + \sum_{j \in N: j<i,\ \sigma^{-1}(j)>\sigma^{-1}(i)} \alpha_j (p_i - x_i). && (6.10)\end{aligned}$$

Now adding expressions (6.6) and (6.10) yields

$$\begin{aligned}& \sum_{j \in N: j<i,\ \sigma^{-1}(j)>\sigma^{-1}(i)} (\alpha_i x_j - \alpha_j p_i) + \sum_{j \in N: j<i,\ \sigma^{-1}(j)>\sigma^{-1}(i)} \alpha_j (p_i - x_i)\\ = & \sum_{j \in N: j<i,\ \sigma^{-1}(j)>\sigma^{-1}(i)} (\alpha_i x_j - \alpha_j x_i)\\ = & \sum_{j \in N: j<i} (\alpha_i x_j - \alpha_j x_i)_+. && (6.11)\end{aligned}$$

The last equality is satisfied because according to the Smith-rule, $\sigma^{-1}(j) > \sigma^{-1}(i)$ implies $\alpha_i x_j - \alpha_j x_i \geq 0$, and $\sigma^{-1}(j) < \sigma^{-1}(i)$ implies $\alpha_i x_j - \alpha_j x_i \leq 0$. Observe that the sum of expressions (6.8) and (6.11) coincides with γ_i. We conclude that

$$\begin{aligned}\delta_i &= \gamma_i - \sum_{j \in N: j > i,\ \sigma^{-1}(j) < \sigma^{-1}(i)} \alpha_j (p_i - x_i) \\ &\quad + \sum_{j \in N: j < i,\ \sigma^{-1}(j) > \sigma^{-1}(i)} \alpha_i (p_j - x_j).\end{aligned}$$

□

Lemma 6.3.4 Let $a_1, a_2, q_1, \bar{q}_1, q_2 \geq 0$ with $q_1 \geq \bar{q}_1$. Then,

$$a_2(q_1 - \bar{q}_1) + (a_2 \bar{q}_1 - a_1 q_2)_+ \geq (a_2 q_1 - a_1 q_2)_+.$$

Proof: If $(a_2 q_1 - a_1 q_2)_+ = 0$, then the inequality is trivially satisfied, so suppose that $(a_2 q_1 - a_1 q_2)_+ > 0$. This implies that $a_2 q_1 - a_1 q_2 > 0$. Straightforwardly it follows that

$$\begin{aligned}a_2(q_1 - \bar{q}_1) + (a_2 \bar{q}_1 - a_1 q_2)_+ &\geq a_2(q_1 - \bar{q}_1) + (a_2 \bar{q}_1 - a_1 q_2) \\ &= a_2 q_1 - a_1 q_2 \\ &= (a_2 q_1 - a_1 q_2)_+.\end{aligned}$$

□

Theorem 6.3.1 Let $(N, \sigma_0, \alpha, \beta, p, \bar{p})$ be a cps situation and (N, v) its corresponding cps game. Let $(\sigma, x) \in AS(N)$ be an optimal processing schedule. Then, $\gamma, \delta \in C(v)$.

Proof: Since γ and δ are efficient by definition, we only show that $\sum_{i \in T} \gamma_i \geq v(T)$ and $\sum_{i \in T} \delta_i \geq v(T)$ for each $T \subseteq N$. Let $T \subseteq N$. We need to show that $\sum_{i \in T} \gamma_i \geq v(T)$ and $\sum_{i \in T} \delta_i \geq v(T)$. We will equivalently show that $\sum_{i \in N \backslash T} \gamma_i + v(T) \leq v(N)$ and $\sum_{i \in N \backslash T} \delta_i + v(T) \leq v(N)$ by constructing a suboptimal processing schedule $(\sigma^{subopt}, p^{subopt}) \in AS(N)$ that depends on an optimal processing schedule of T. We show that the total cost savings obtained at this processing schedule exceed both $\sum_{i \in N \backslash T} \gamma_i + v(T)$ and $\sum_{i \in N \backslash T} \delta_i + v(T)$. Obviously the cost savings at the suboptimal processing schedule are at most $v(N)$.

We first find usable expressions for $\sum_{i\in N\backslash T}\gamma_i$ and $\sum_{i\in N\backslash T}\delta_i$. Note that

$$\begin{aligned}
\sum_{i\in N\backslash T}\gamma_i &= \sum_{i\in N\backslash T}\Big(\sum_{j\in N:j\geq i}\alpha_j-\beta_i\Big)(p_i-x_i) && (6.12)\\
&\quad+\sum_{i,j\in N\backslash T:j<i}(\alpha_i x_j-\alpha_j x_i)_+ && (6.13)\\
&\quad+\sum_{j\in T,i\in N\backslash T:j<i}(\alpha_i x_j-\alpha_j x_i)_+ && (6.14)
\end{aligned}$$

and that

$$\begin{aligned}
\sum_{i\in N\backslash T}\delta_i &= \sum_{i\in N\backslash T}\gamma_i\\
&\quad-\sum_{i\in N\backslash T}\quad\sum_{j\in N:j>i,\sigma^{-1}(j)<\sigma^{-1}(i)}\alpha_j(p_i-x_i) && (6.15)\\
&\quad+\sum_{i\in N\backslash T}\quad\sum_{j\in N:j<i,\sigma^{-1}(j)>\sigma^{-1}(i)}\alpha_i(p_j-x_j), && (6.16)\\
&= \sum_{i\in N\backslash T}\gamma_i && (6.17)\\
&\quad-\sum_{i\in N\backslash T}\quad\sum_{j\in T:j>i,\sigma^{-1}(j)<\sigma^{-1}(i)}\alpha_j(p_i-x_i) && (6.18)\\
&\quad+\sum_{i\in N\backslash T}\quad\sum_{j\in T:j<i,\sigma^{-1}(j)>\sigma^{-1}(i)}\alpha_i(p_j-x_j), && (6.19)
\end{aligned}$$

where the first equality holds because of Lemma 6.3.3. The second equality is satisfied because for each pair $i,j\in N\backslash T$ with $j>i$ and $\sigma^{-1}(j)<\sigma^{-1}(i)$ every term in (6.15) also appears in (6.16) but with opposite sign.

Let $(\sigma^T,p^T)\in AS(T)$ be an optimal processing schedule for coalition T. Consider the processing schedule $(\sigma^{subopt},p^{subopt})\in AS(N)$, where $p_j^{subopt}=p_j^T$ if $j\in T$, $p_j^{subopt}=x_j$ if $j\in N\backslash T$, and σ^{subopt} is the order obtained by applying the Smith-rule using the suboptimal processing times. The total cost savings for the grand coalition at this processing schedule equal

$$P = \sum_{j\in T}\Big(\sum_{i\in T:i\geq j}\alpha_i-\beta_j\Big)(p_j-p_j^{subopt}) \qquad (6.20)$$

$$+\sum_{j\in T}\Bigg(\sum_{i\in N\backslash T:i>j}\alpha_i\Bigg)(p_j-p_j^{subopt}) \tag{6.21}$$

$$+\sum_{j\in N\backslash T}\Bigg(\sum_{i\in N:i\geq j}\alpha_i-\beta_j\Bigg)(p_j-p_j^{subopt}) \tag{6.22}$$

$$+\sum_{j,i\in N\backslash T:j<i}(\alpha_i p_j^{subopt}-\alpha_j p_i^{subopt})_+ \tag{6.23}$$

$$+\sum_{j\in N\backslash T,i\in T:j<i}(\alpha_i p_j^{subopt}-\alpha_j p_i^{subopt})_+ \tag{6.24}$$

$$+\sum_{j\in T,i\in N\backslash T:j<i}(\alpha_i p_j^{subopt}-\alpha_j p_i^{subopt})_+ \tag{6.25}$$

$$+\sum_{j,i\in T:j<i}(\alpha_i p_j^{subopt}-\alpha_j p_i^{subopt})_+. \tag{6.26}$$

Expressions (6.20) and (6.21) are the cost savings obtained by crashing the jobs of T, and expression (6.22) the cost savings obtained by crashing the jobs of $N\backslash T$. The cost savings obtained by rearranging the jobs are equal to the sum of expressions (6.23), (6.24), (6.25) and (6.26).

Now we will show that the sum of (6.12), (6.13), (6.14), (6.19) and $v(T)$ is exceeded by P. Since (6.19) is non-negative, this shows that $\sum_{i\in N\backslash T}\gamma_i+v(T)$ is exceeded by P. Furthermore, since (6.18) is non-positive, it also shows that $\sum_{i\in N\backslash T}\delta_i+v(T)$ is exceeded by P.

First note that the sum of expressions (6.20) and (6.26) exceeds $v(T)$ because $p_i^{subopt}=p_i^T$ for all $i\in T$. It also holds that (6.22) coincides with (6.12) as well as (6.23) coincides with (6.13) because $p_j^{subopt}=x_j$ for all $j\in N\backslash T$. Finally, note that expression (6.24) is non-negative. Hence, for showing that the sum of expressions (6.12), (6.13), (6.14), (6.19) and $v(T)$ is exceeded by P it is sufficient to show that the sum of (6.14) and (6.19) is exceeded by the sum of (6.21) and (6.25). We will show this by comparing these sums for each pair $i,j\in N$ with $j\in T$, $i\in N\backslash T$ and $j<i$.

Let $j\in T$ and $i\in N\backslash T$ be such that $j<i$. Note that $p_i^{subopt}=x_i$ and that $p_j^{subopt}=p_j^T$. We distinguish between two cases.

`Case 1:` $\sigma^{-1}(j)\leq\sigma^{-1}(i)$.

In this case, i and j do not have a corresponding term in (6.19). Therefore we only need to compare the term in (6.14) corresponding to i and j with the terms in (6.21) and (6.25) corresponding to i and j.

If $p_j^{subopt} \geq x_j$, then $(\alpha_i p_j^{subopt} - \alpha_j p_i^{subopt})_+ = (\alpha_i p_j^{subopt} - \alpha_j x_i)_+ \geq (\alpha_i x_j - \alpha_j x_i)_+$. Hence, the term in (6.14) corresponding to i and j is exceeded by the term in (6.25) corresponding to i and j. Since the term in (6.21) corresponding to i and j is non-negative we conclude that the term in (6.14) corresponding to i and j is exceeded by the sum of the terms in (6.21) and (6.25) corresponding to i and j.

So assume that $p_j^{subopt} < x_j$. Since optimal processing times only take two values by assumption, $p_j^{subopt} = \bar{p}_j$ and $x_j = p_j$. It is easily verified, using Lemma 6.3.4 with $a_1 = \alpha_j$, $a_2 = \alpha_i$, $q_1 = p_j$, $\bar{q}_1 = p_j^{subopt}$ and $q_2 = p_i^{subopt}$, that $\alpha_i(p_j - p_j^{subopt}) + (\alpha_i p_j^{subopt} - \alpha_j p_i^{subopt})_+ \geq (\alpha_i p_j - \alpha_j p_i^{subopt})_+ = (\alpha_i x_j - \alpha_j x_i)_+$. Here the equality holds because $p_j = x_j$ and $p_i^{subopt} = x_i$.

We conclude that the terms in (6.21) and (6.25) corresponding to i and j exceed the term in (6.14) corresponding to i and j.

`Case 2:` $\sigma^{-1}(j) > \sigma^{-1}(i)$.

Since $\sigma^{-1}(j) > \sigma^{-1}(i)$ we conclude, using the Smith-rule, that the urgency index of i exceeds the urgency index of j. That is, $\frac{\alpha_i}{x_i} \geq \frac{\alpha_j}{x_j}$ and therefore, $\alpha_i x_j - \alpha_j x_i \geq 0$. Straightforwardly we obtain

$$\begin{aligned}
& \alpha_i(p_j - p_j^{subopt}) + (\alpha_i p_j^{subopt} - \alpha_j p_i^{subopt})_+ \\
\geq \; & \alpha_i(p_j - p_j^{subopt}) + (\alpha_i p_j^{subopt} - \alpha_j p_i^{subopt}) \\
= \; & \alpha_i p_j - \alpha_j p_i^{subopt} \\
= \; & (\alpha_i x_j - \alpha_j p_i^{subopt}) + \alpha_i(p_j - x_j) \\
= \; & (\alpha_i x_j - \alpha_j x_i)_+ + \alpha_i(p_j - x_j),
\end{aligned}$$

where the last equality holds since $p_i^{subopt} = x_i$ and $\alpha_i x_j - \alpha_j x_i \geq 0$. We conclude that the terms in expressions (6.21) and (6.25) corresponding to i and j exceed the terms in expressions (6.14) and (6.19) corresponding to i and j. $\square$

In the upcoming part of this section we study marginal vectors that provide core elements. In the next theorem we show that many marginal vectors are core elements by showing that the corresponding orders are permutationally convex. In particular we show that any order with players $2, \ldots, |N|$ ordered backwards is permutationally convex. First we need a lemma. This lemma states that if a coalition consists of several components, then each optimal processing schedule of this coalition is also optimal for its last component. The intuition behind this result is that the jobs from other components do not benefit from the crashing of jobs in the last component.

Lemma 6.3.5 Let $(N, \sigma_0, \alpha, \beta, p, \bar{p})$ be a cps situation. Let $S \subseteq N$ and let $T \in \mathcal{C}(S)$ be the last component of S. Then every optimal schedule for S restricted to T is optimal for T.

The proof is omitted since it is trivial.

Theorem 6.3.2 Let $(N, \sigma_0, \alpha, \beta, p, \bar{p})$ be a cps situation and (N, v) its corresponding cps game. Let $j \in \{1, \ldots, |N|\}$ and let $\sigma \in \Pi(N)$ be such that $\sigma(i) = |N| + 1 - i$ for all $i \in \{1, \ldots, j-1\}$, $\sigma(i) = |N| + 2 - i$ for all $i \in \{j+1, \ldots, |N|\}$ and $\sigma(j) = 1$. Then, σ is permutationally convex for (N, v). In particular, $m^{\sigma}(v) \in C(v)$.

Proof: We need to show that for all $i, k \in \{0, \ldots, |N|-1\}$ with $i < k$ and all $S \subseteq N \backslash [\sigma(k), \sigma]$ with $S \neq \emptyset$ that expression (2.1) is satisfied.

So let $i, k \in \{0, \ldots, |N|-1\}$ with $i < k$ and $S \subseteq N \backslash [\sigma(k), \sigma]$ with $S \neq \emptyset$. We assume that $i \geq 1$, because for $i = 0$ expression (2.1) is satisfied due to superadditivity. For notational convenience, let $I = [\sigma(i), \sigma]$ and $K = [\sigma(k), \sigma]$. Observe that (2.1) now boils down to

$$v(I \cup S) + v(K) \leq v(K \cup S) + v(I). \tag{6.27}$$

Because of the structure of σ, I and K both consist of at most two connected components. In particular, they consist of possibly player 1 and a tail[6] of σ_0. If $1 \in K$, then we denote the first component by $K_1 = \{1\}$ and the second by $K_2 = K \backslash \{1\}$. If $1 \notin K$, then there is one component which we denote by

[6]A tail of σ_0 is a coalition $T \subseteq N$ such that $T = \{\sigma_0(j), \ldots, \sigma_0(|N|)\}$ for some $j \in \{1, \ldots, |N|\}$.

K_2 and we define $K_1 = \emptyset$. Similarly, if $1 \in I$, then $I_1 = \{1\}$ and $I_2 = I\backslash\{1\}$. We remark that it can occur that $I_2 = \emptyset$, in case $\sigma = (1, |N|, |N|-1, \ldots, 2)$ and $i = 1$. If $1 \notin I$, then there is one component which we denote by I_2 and we define $I_1 = \emptyset$.

Let $(\tau^{I\cup S}, p^{I\cup S})$ be an optimal processing schedule of coalition $I \cup S$, and (τ^K, p^K) an optimal processing schedule of coalition K. We will create suboptimal processing schedules for coalitions $K \cup S$ and I, depending on $(\tau^{I\cup S}, p^{I\cup S})$ and (τ^K, p^K). In particular we "allocate" the processing times of coalitions $I \cup S$ and K to coalitions $K \cup S$ and I. By applying the Smith-rule we then obtain processing schedules that give lower bounds for $v(K\cup S)$ and $v(I)$. We will show that the sum of these lower bounds exceeds the sum of $v(I \cup S)$ and $v(K)$. We distinguish between two cases.

`Case 1:` $I_2 \subsetneq K_2$.

Before we obtain our suboptimal processing schedules, we first derive expressions for $v(K)$ and $v(I \cup S)$. Observe that

$$
\begin{aligned}
v(K) \quad &= \quad \sum_{l\in K_1}\Bigg(\sum_{m\in K}\alpha_m - \beta_l\Bigg)(p_l - p_l^K) &(6.28)\\
&\quad\quad + v(K_2). &(6.29)
\end{aligned}
$$

The cost savings of coalition K can be divided into two parts. The cost savings obtained by a possible crash of job 1 equal (6.28). Note that if $1 \notin K$, then $K_1 = \emptyset$ and expression (6.28) is zero by definition. The other cost savings are obtained by interchanging and crashing jobs of K_2. These cost savings equal (6.29) according to Lemma 6.3.5.

Since K_2 is a tail of σ_0, $I_2 \subsetneq K_2$ and $K_2 \cap S = \emptyset$, we conclude that S is not connected to I_2. Now let $S_1, \ldots, S_t$, $t \geq 1$, be the components of $(I \cup S)\backslash I_2$. We assume that if $1 \in I$, then $1 \in S_1$. Now

$$
\begin{aligned}
v(I\cup S) \quad &= \quad \sum_{l\in I_1}\Bigg(\sum_{m\in I\cup S}\alpha_m - \beta_l\Bigg)(p_l - p_l^{I\cup S}) &(6.30)\\
&\quad\quad + \sum_{l\in S}\Bigg(\sum_{m\in I_2\cup S: m\geq l}\alpha_m - \beta_l\Bigg)(p_l - p_l^{I\cup S}) &(6.31)
\end{aligned}
$$

$$+ \sum_{l,m \in S_1 : l<m} (\alpha_m p_l^{I\cup S} - \alpha_l p_m^{I\cup S})_+ \tag{6.32}$$

$$+ \sum_{a=2}^{t} \sum_{l,m \in S_a : l<m} (\alpha_m p_l^{I\cup S} - \alpha_l p_m^{I\cup S})_+ \tag{6.33}$$

$$+ v(I_2). \tag{6.34}$$

If $1 \in I$, then expression (6.30) denotes the cost savings obtained by a possible crash of job 1. The cost savings obtained by crashing the jobs in S equal (6.31). Expressions (6.32) and (6.33) are the cost savings obtained by switching the jobs in $I_1 \cup S$. Because of Lemma 6.3.5 the cost savings obtained by crashing and rearranging jobs in I_2 can be expressed as (6.34).

Now we will create suboptimal processing schedules (π^I, p^I) and $(\pi^{K\cup S}, p^{K\cup S})$ for coalitions I and $K \cup S$, respectively. Let

$$p_l^I = \begin{cases} p_l^{I\cup S}, & \text{if } l \in I_2; \\ \max\{p_l^{I\cup S}, p_l^K\}, & \text{if } l = 1; \\ p_l, & \text{if } l \in N\backslash I,\ l \neq 1. \end{cases}$$

Note that if $1 \notin I$, then $1 \notin K$ or $1 \notin I \cup S$. So if $1 \notin I$, then $p_1^K = p_1$ or $p_1^{I\cup S} = p_1$. Therefore, $p_1^I = p_1$. We conclude that p^I satisfies admissibility constraints (6.2) and (6.3).

Furthermore let π^I be obtained from σ_0 by rearranging the jobs of I according to the Smith-rule using processing times p^I, taking of course into account admissibility constraint (6.1). This processing schedule restricted to I_2 is an optimal processing schedule for I_2 according to Lemma 6.3.5. This yields

$$v(I) \geq \sum_{l \in I_1} \left(\sum_{m \in I} \alpha_m - \beta_l \right) (p_l - p_l^I) \tag{6.35}$$

$$+ v(I_2). \tag{6.36}$$

Similarly, let

$$p_l^{K\cup S} = \begin{cases} p_l^K, & \text{if } l \in K_2; \\ p_l^{I\cup S}, & \text{if } l \in S,\ l \neq 1; \\ \min\{p_l^{I\cup S}, p_l^K\}, & \text{if } l = 1; \\ p_l, & \text{if } l \in N\backslash(K \cup S),\ l \neq 1. \end{cases}$$

Observe that if $1 \notin K \cup S$, then $1 \notin I \cup S$ and $1 \notin K$. Hence, if $1 \notin K \cup S$, then $p_1^K = p_1$, $p_1^{I \cup S} = p_1$ and $p_1^{K \cup S} = p_1$. We conclude that $p^{K \cup S}$ satisfies admissibility constraints (6.2) and (6.3).

Now using these processing times, let $\pi^{K \cup S}$ be the order obtained from σ_0 by interchanging the jobs according to the Smith-rule, while of course taking into account admissibility constraint (6.1). However, only interchange two jobs if both jobs are in K_2, or both jobs are in $I_1 \cup S$. This last condition is a technical detail in order to keep the number of terms of our lower bound for the cost savings of coalition $K \cup S$ more manageable. Observe that this restriction only lowers our lower bound of $v(K \cup S)$. Again note, by Lemma 6.3.5, that this processing schedule restricted to K_2 is optimal for K_2. This yields

$$\begin{aligned}
v(K \cup S) \quad \geq \quad & \sum_{l \in K_1} \Bigg(\sum_{m \in K \cup S} \alpha_m - \beta_l \Bigg) (p_l - p_l^{K \cup S}) && (6.37) \\
& + \sum_{l \in S} \Bigg(\sum_{m \in K_2 \cup S : m \geq l} \alpha_m - \beta_l \Bigg) (p_l - p_l^{K \cup S}) && (6.38) \\
& + \sum_{l,m \in S_1 : l < m} (\alpha_m p_l^{K \cup S} - \alpha_l p_m^{K \cup S})_+ && (6.39) \\
& + \sum_{a=2}^{t} \sum_{l,m \in S_a : l < m} (\alpha_m p_l^{K \cup S} - \alpha_l p_m^{K \cup S})_+ && (6.40) \\
& + v(K_2). && (6.41)
\end{aligned}$$

Now first observe that expressions (6.36) and (6.34) coincide, as well as expressions (6.41) and (6.29). Furthermore expression (6.38) exceeds expression (6.31) since $I_2 \subseteq K_2$ and $p_l^{K \cup S} = p_l^{I \cup S}$ for all $l \in S$. It also holds that expression (6.33) coincides with expression (6.40) because $p_l^{K \cup S} = p_l^{I \cup S}$ for all $l \in S$. So showing (6.27) is satisfied boils down to showing that the sum of expressions (6.35), (6.37) and (6.39) exceeds the sum of expressions (6.28), (6.30) and (6.32). We now distinguish between three subcases.

`Subcase 1a:` $1 \notin K$.

This implies that $K_1 = \emptyset$, and thus that $I_1 = \emptyset$. Therefore, expressions (6.28), (6.30), (6.35) and (6.37) all are equal to zero. Hence, it is sufficient to show that (6.39) exceeds (6.32). Because $1 \notin K$, it follows that $p_1^k = p_1$. By definition of $p_1^{K\cup S}$ it is now satisfied that $p_1^{K\cup S} = p_1^{I\cup S}$. Therefore, $p_j^{I\cup S} = p_j^{K\cup S}$ for all $j \in S_1$. We conclude that (6.39) and (6.32) coincide.

`Subcase 1b:` $1 \in K$ and $p_1^{K\cup S} = p_1^{I\cup S}$.

Because $p_1^{K\cup S} = p_1^{I\cup S}$, it follows from the definition of $p_1^{K\cup S}$ that $p_1^{I\cup S} \leq p_1^K$. We conclude that $p_1^I = p_1^K$. Note that (6.39) and (6.32) coincide, since $p_j^{I\cup S} = p_j^{K\cup S}$ for all $j \in S_1$. So showing that (6.27) is satisfied, boils down to showing that the sum of expressions (6.35) and (6.37) exceeds the sum of expressions (6.28) and (6.30).

Now first suppose that $p_1^K = p_1$. Then, expression (6.28) is equal to zero. Since $p_1^I = p_1^K$ it follows that expression (6.35) is equal to zero as well. Now if $I_1 \neq \emptyset$, then (6.37) exceeds expression (6.30), since $p_1^{K\cup S} = p_1^{I\cup S}$ and $I \subseteq K$. So assume that $I_1 = \emptyset$. In this case expression (6.30) is equal to zero, so it remains to show that expression (6.37) is non-negative. Because $1 \in K$, it follows that $1 \notin S$, and therefore $1 \notin I \cup S$. So $p_1^{I\cup S} = p_1$, and therefore $p_1^{K\cup S} = p_1$ as well. Hence, expression (6.37) is equal to zero. We conclude that (6.27) is satisfied.

Secondly, suppose that $p_1^K < p_1$. Since optimal processing times can only take two values we conclude that $p_1^K = \bar{p}_1$. Since by assumption of subcase 1b, $p_1^{K\cup S} = \min\{p_1^K, p_1^{I\cup S}\} = p_1^{I\cup S}$, it follows that $p_1^{I\cup S} = \bar{p}_1$. We conclude that $p_1^I = \max\{p_1^K, p_1^{I\cup S}\} = \max\{\bar{p}_1, \bar{p}_1\} = \bar{p}_1$.

Observe that because $1 \in K$, that $1 \notin S$. Since $p_1^{I\cup S} < p_1$, this implies that $1 \in I$. Thus, $K_1 = I_1 = \{1\}$. Hence, it follows for the sum of expressions (6.28) and (6.30) that

$$\begin{aligned}
&\sum_{l\in K_1}\Bigg(\sum_{m\in K}\alpha_m - \beta_l\Bigg)(p_l - p_l^K) + \sum_{l\in I_1}\Bigg(\sum_{m\in I\cup S}\alpha_m - \beta_l\Bigg)(p_l - p_l^{I\cup S}) \\
= \ &\Bigg(\sum_{m\in K}\alpha_m - \beta_1\Bigg)(p_1 - \bar{p}_1) + \Bigg(\sum_{m\in I\cup S}\alpha_m - \beta_1\Bigg)(p_1 - \bar{p}_1) \\
= \ &\Bigg(\sum_{m\in K\cup S}\alpha_m - \beta_1\Bigg)(p_1 - \bar{p}_1) + \Bigg(\sum_{m\in I}\alpha_m - \beta_1\Bigg)(p_1 - \bar{p}_1),
\end{aligned}$$

where the first equality is satisfied since $K_1 = I_1 = \{1\}$ and $p_1^K = p_1^{I\cup S} = \bar{p}_1$. Note that this last expression is equal to the sum of expressions (6.35) and

(6.37) since $p_1^I = p_1^{K\cup S} = \bar{p}_1$ and $K_1 = I_1 = \{1\}$. We conclude that (6.27) is satisfied.

`Subcase 1c:` $1 \in K$ and $p_1^{K\cup S} < p_1^{I\cup S}$.

We now necessarily have that $p_1^{K\cup S} = p_1^K$ and that $p_1^I = p_1^{I\cup S}$. Since optimal processing times can only take two values it follows that $p_1^{K\cup S} = p_1^K = \bar{p}_1$ and that $p_1^I = p_1^{I\cup S} = p_1$. Therefore expressions (6.30) and (6.35) are both equal to zero. So showing that (6.27) is satisfied boils down to showing that the sum of expressions (6.37) and (6.39) exceeds the sum of expressions (6.28) and (6.32). First observe for expression (6.37) that

$$\begin{aligned}
&\sum_{l\in K_1}\left(\sum_{m\in K\cup S}\alpha_m - \beta_l\right)(p_l - p_l^{K\cup S}) \\
= &\left(\sum_{m\in K\cup S}\alpha_m - \beta_1\right)(p_1 - \bar{p}_1) \\
\geq &\left(\sum_{m\in K}\alpha_m - \beta_1\right)(p_1 - \bar{p}_1) && (6.42)\\
&+\left(\sum_{m\in S_1\backslash\{1\}}\alpha_m\right)(p_1 - \bar{p}_1), && (6.43)
\end{aligned}$$

where the equality holds because $K_1 = \{1\}$ and because $p_1^{K\cup S} = \bar{p}_1$. The inequality holds since $(S_1\backslash\{1\}) \subseteq S$. Since $K_1 = \{1\}$ and $p_1^K = \bar{p}_1$, it follows that expression (6.42) coincides with expression (6.28). Therefore it is now sufficient to show that the sum of expressions (6.39) and (6.43) exceeds expression (6.32). If $1 \notin S_1$, then expression (6.39) coincides with expression (6.32) since $p_m^{K\cup S} = p_m^{I\cup S}$ for all $m \in S_1\backslash\{1\}$. The non-negativity of expression (6.43) now implies that the sum of expressions (6.39) and (6.43) exceeds expression (6.32). So assume that $1 \in S_1$, and rewrite expression (6.39) as

$$\begin{aligned}
&\sum_{l,m\in S_1:l<m}(\alpha_m p_l^{K\cup S} - \alpha_l p_m^{K\cup S})_+ \\
= &\sum_{l,m\in S_1\backslash\{1\}:l<m}(\alpha_m p_l^{K\cup S} - \alpha_l p_m^{K\cup S})_+ && (6.44)\\
&+\sum_{m\in S_1\backslash\{1\}}(\alpha_m \bar{p}_1 - \alpha_1 p_m^{K\cup S})_+. && (6.45)
\end{aligned}$$

For the equality we have used that $p_1^{K\cup S} = \bar{p}_1$. For the sum of expressions (6.43) and (6.45) we have that

$$\begin{aligned}
& \Big(\sum_{m\in S_1\backslash\{1\}} \alpha_m\Big)(p_1 - \bar{p}_1) + \sum_{m\in S_1\backslash\{1\}} (\alpha_m \bar{p}_1 - \alpha_1 p_m^{K\cup S})_+ \\
\geq & \sum_{m\in S_1\backslash\{1\}} (\alpha_m p_1 - \alpha_1 p_m^{K\cup S})_+ \\
= & \sum_{m\in S_1\backslash\{1\}} (\alpha_m p_1^{I\cup S} - \alpha_1 p_m^{I\cup S})_+. \qquad (6.46)
\end{aligned}$$

The inequality holds because of Lemma 6.3.4 by taking $a_1 = \alpha_1$, $a_2 = \alpha_m$, $q_1 = p_1$, $\bar{q}_1 = \bar{p}_1$ and $q_2 = p_m^{K\cup S}$. The equality is satisfied because $p_1^{I\cup S} = p_1$ and $p_m^{K\cup S} = p_m^{I\cup S}$ for all $m \in S_1\backslash\{1\}$. Now observe that the sum of expressions (6.44) and (6.46) coincides with expression (6.32) since $p_m^{K\cup S} = p_m^{I\cup S}$ for all $m \in S_1\backslash\{1\}$ and our assumption that $1 \in S_1$. We conclude that (6.27) is satisfied.

`Case 2:` $I_2 = K_2$.

Since we have assumed that $i < k$ and hence that $I \neq K$ we conclude, using the structure of σ, that $I_1 = \emptyset$ and $K_1 = \{1\}$. We have

$$\begin{aligned}
v(K) \quad = \quad & \Big(\sum_{m\in K} \alpha_m - \beta_1\Big)(p_1 - p_1^K) \qquad (6.47) \\
& + v(K_2) \qquad (6.48)
\end{aligned}$$

and

$$\begin{aligned}
v(I\cup S) \quad = \quad & \sum_{l\in I\cup S}\Big(\sum_{m\in I\cup S: m\geq l} \alpha_m - \beta_l\Big)(p_l - p_l^{I\cup S}) \qquad (6.49) \\
& + \sum_{T\in \mathcal{C}(I\cup S)} \sum_{l,m\in T: l<m} (\alpha_m p_l^{I\cup S} - \alpha_l p_m^{I\cup S})_+. \qquad (6.50)
\end{aligned}$$

First we note, because $I_1 = \emptyset$, that $v(I) = v(I_2) = v(K_2)$. So showing that (6.27) is satisfied boils down to showing that the sum of expressions (6.47), (6.49) and (6.50) is exceeded by $v(K\cup S)$. We will obtain a lower bound of $v(K\cup S)$ by creating a processing schedule for coalition $K\cup S$, which we will denote by $(\pi^{K\cup S}, p^{K\cup S})$. Let the processing times be given by

$$p_l^{K\cup S} = \begin{cases} p_l^{I\cup S}, & \text{if } l \in I \cup S; \\ p_l^K, & \text{if } l = 1; \\ p_l, & \text{if } l \in N\backslash(K \cup S). \end{cases}$$

Let $\pi^{K\cup S}$ be constructed by rearranging the jobs of coalition $K \cup S$ using the Smith-rule and our suboptimal processing times. However, only switch jobs if both are in $I \cup S$. Observe that $(p^{K\cup S}, \pi^{K\cup S}) \in AS(K \cup S)$. We can conclude for the cost savings of $K \cup S$ that

$$\begin{aligned} v(K \cup S) \quad \geq \quad & \left(\sum_{m\in K\cup S} \alpha_m - \beta_1\right)(p_1 - p_1^{K\cup S}) & (6.51) \\ & + \sum_{l\in(K\cup S)\backslash\{1\}} \left(\sum_{m\in K\cup S: m\geq l} \alpha_m - \beta_l\right)(p_l - p_l^{K\cup S}) & (6.52) \\ & + \sum_{T\in\mathcal{C}(I\cup S)} \sum_{l,m\in T: l<m} (\alpha_m p_l^{K\cup S} - \alpha_l p_m^{K\cup S})_+. & (6.53) \end{aligned}$$

Expression (6.47) is exceeded by expression (6.51) since $p_1^{K\cup S} = p_1^K$. Note that expression (6.52) coincides with expression (6.49) since $(K \cup S)\backslash\{1\} = I \cup S$ and because $p_l^{K\cup S} = p_l^{I\cup S}$ for all $l \in I \cup S$. Furthermore we have that expressions (6.50) and (6.53) coincide because $p_l^{I\cup S} = p_l^{K\cup S}$ for all $l \in I \cup S$. We conclude that the sum of $v(K \cup S)$ and $v(I)$ exceeds the sum of $v(I \cup S)$ and $v(K)$. $\square$

The orders described in Theorem 6.3.2 are not the only permutationally convex orders of cps games. Using Corollary 2.5.1, we conclude that if $\sigma \in \Pi(N)$ is permutationally convex, then $\sigma_{|N|-1}$ is permutationally convex as well. Moreover, we will show for cps games that if $\sigma \in \Pi(N)$ is permutationally convex, then σ_1 is permutationally convex as well. We first need the following lemma.

Lemma 6.3.6 Let $(N, \sigma_0, \alpha, \beta, p, \bar{p})$ be a cps situation and (N, v) its corresponding cps game. Let $S, T \subseteq N$ with $|S \cap T| = 1$. Then, $v(S) + v(T) \leq v(S \cup T)$.

Proof: Let $S, T \subseteq N$ and $i \in N$ be such that $S \cap T = \{i\}$. Let $T^* \in \mathcal{C}(T)$ be such that $i \in T^*$. Define $T_l^* = \{m \in T^* : m < i\}$ be those players in T^* on the left-hand side of i, and $T_r^* = \{m \in T^* : m > i\}$ be those players in T^* on the right-hand side of i. Let $(\sigma^S, p^S) \in AS(S)$ be an optimal processing schedule for coalition S and $(\sigma^T, p^T) \in AS(T)$ be an optimal processing schedule for coalition T. Then

$$\begin{aligned} v(S) &= \sum_{j \in S} \Big(\sum_{k \in S : k \geq j} \alpha_k - \beta_j \Big)(p_j - p_j^S) && (6.54)\\ &\quad + \sum_{U \in \mathcal{C}(S)} \sum_{l,m \in U : l < m} (\alpha_m p_l^S - \alpha_l p_m^S)_+. && (6.55) \end{aligned}$$

The cost savings for coalition S can be split into two parts. The first part, (6.54), are the (possibly negative) cost savings that are obtained by crashing jobs. The second part of the cost savings, displayed in expression (6.55), are the cost savings obtained by interchanging jobs. Similarly,

$$\begin{aligned} v(T) &= \sum_{j \in T} \Big(\sum_{k \in T : k \geq j} \alpha_k - \beta_j \Big)(p_j - p_j^T) && (6.56)\\ &\quad + \sum_{U \in \mathcal{C}(T)} \sum_{l,m \in U : l < m} (\alpha_m p_l^T - \alpha_l p_m^T)_+. && (6.57) \end{aligned}$$

We will now construct a suboptimal processing schedule for coalition $S \cup T$. We will show that the cost savings obtained at this processing schedule exceed the sum of $v(S)$ and $v(T)$. Consider the processing schedule $(\sigma^{S \cup T}, p^{S \cup T}) \in AS(S \cup T)$ for coalition $S \cup T$, where

$$p_j^{S \cup T} = \begin{cases} p_j^S, & \text{if } j \in S \backslash \{i\};\\ p_j^T, & \text{if } j \in T \backslash \{i\};\\ \min\{p_j^S, p_j^T\}, & \text{if } j = i;\\ p_j, & \text{if } j \in N \backslash (S \cup T). \end{cases}$$

Note that $p^{S \cup T}$ satisfies admissibility constraints (6.2) and (6.3). Furthermore, let $\sigma^{S \cup T}$ be the order obtained by rearranging the jobs of $S \cup T$ according to the Smith-rule using as processing times $p^{S \cup T}$ and taking into account admissibility constraint (6.1). This yields

$$\begin{aligned}
v(S\cup T) \;\geq\; & \sum_{j\in S\cup T}\Big(\sum_{k\in S\cup T:k\geq j}\alpha_k-\beta_j\Big)(p_j-p_j^{S\cup T}) \\
& +\sum_{U\in\mathcal{C}(S\cup T)}\sum_{l,m\in U:l<m}(\alpha_m p_l^{S\cup T}-\alpha_l p_m^{S\cup T})_+ \\
= \; & \sum_{j\in S\setminus\{i\}}\Big(\sum_{k\in S\cup T:k\geq j}\alpha_k-\beta_j\Big)(p_j-p_j^{S}) && (6.58)\\
& +\sum_{j\in T\setminus\{i\}}\Big(\sum_{k\in S\cup T:k\geq j}\alpha_k-\beta_j\Big)(p_j-p_j^{T}) && (6.59)\\
& +\Big(\sum_{k\in S\cup T:k\geq i}\alpha_k-\beta_i\Big)(p_i-p_i^{S\cup T}) && (6.60)\\
& +\sum_{U\in\mathcal{C}(S\cup T)}\sum_{l,m\in U:l<m}(\alpha_m p_l^{S\cup T}-\alpha_l p_m^{S\cup T})_+. && (6.61)
\end{aligned}$$

We distinguish between three cases in order to prove our inequality.

`Case 1:` $p_i^S=p_i^T=p_i$.

First note that $p_j^{S\cup T}=p_j^S$ for all $j\in S$ and that $p_j^{S\cup T}=p_j^T$ for all $j\in T$. Therefore, expression (6.60) is equal to zero because $p_i^{S\cup T}=p_i$. Furthermore, expression (6.58) exceeds expression (6.54), as well as expression (6.59) exceeds expression (6.56), since $p_i^S=p_i^T=p_i$. So showing that $v(S)+v(T)\leq v(S\cup T)$ boils down to showing that expression (6.61) exceeds the sum of expressions (6.55) and (6.57). Let $U\in\mathcal{C}(S)$ and $j,h\in U$ with $j<h$. It follows that $j,h\in W$ for some $W\in\mathcal{C}(S\cup T)$. We conclude that the term in (6.55) dealing with j and h also appears in (6.61). Similarly, for each $U\in\mathcal{C}(T)$ and each pair $j,h\in U$ with $j<h$, there is a $W\in\mathcal{C}(S\cup T)$ with $j,h\in W$. Therefore, the term in (6.57) dealing with j,h also appears in (6.61). Observe that each pair in S is not in T, and that each pair in T is not in S, because $|S\cap T|=1$. We conclude, due to the non-negativity of every term in (6.61), that (6.61) exceeds the sum of (6.55) and (6.57).

`Case 2:` $p_i^S=p_i^T=\bar{p}_i$.

Note, by definition of $p_i^{S\cup T}$, that $p_i^{S\cup T}=p_i^S=p_i^T$. This implies that $p_j^{S\cup T}=p_j^S$ for all $j\in S$ and $p_j^{S\cup T}=p_j^T$ for all $j\in T$. First we develop a lower bound for (6.60):

$$
\begin{aligned}
& \Big(\sum_{k\in S\cup T:k\geq i}\alpha_k-\beta_i\Big)(p_i-p_i^{S\cup T}) \\
= & \Big(\sum_{k\in S:k\geq i}\alpha_k-\beta_i\Big)(p_i-p_i^{S\cup T}) \\
& +\Big(\sum_{k\in T:k\geq i}\alpha_k-\beta_i\Big)(p_i-p_i^{S\cup T})+(\beta_i-\alpha_i)(p_i-p_i^{S\cup T}) \\
\geq & \Big(\sum_{k\in S:k\geq i}\alpha_k-\beta_i\Big)(p_i-p_i^{S}) && (6.62)\\
& +\Big(\sum_{k\in T:k\geq i}\alpha_k-\beta_i\Big)(p_i-p_i^{T}) && (6.63)
\end{aligned}
$$

where the inequality is satisfied since $\beta_i \geq \alpha_i$, and by assumption $p_i^{S\cup T} = p_i^S = p_i^T$. Observe that the sum of expressions (6.58) and (6.62) exceeds expression (6.54) and that the sum of expressions (6.59) and (6.63) exceeds expression (6.56). Hence, showing that $v(S)+v(T)\leq v(S\cup T)$ boils down to showing that expression (6.61) exceeds the sum of expressions (6.55) and (6.57). For this last statement we refer to Case 1, where we already showed this inequality. Note that we can refer to Case 1, since in both cases we have that $p_j^{S\cup T} = p_j^S$ for all $j \in S$ and $p_j^{S\cup T} = p_j^T$ for all $j \in T$.

`Case 3:` $p_i^S \neq p_i^T$.

Without loss of generality assume that $p_i^S < p_i^T$, or equivalently $p_i^S = \bar{p}_i$ and $p_i^T = p_i$. Hence, by definition of $p_i^{S\cup T}$, $p_i^{S\cup T} = p_i^S = \bar{p}_i$. First we derive a lower bound for (6.60):

$$
\begin{aligned}
& \Big(\sum_{k\in S\cup T:k\geq i}\alpha_k-\beta_i\Big)(p_i-p_i^{S\cup T}) \\
= & \Big(\sum_{k\in S:k\geq i}\alpha_k-\beta_i\Big)(p_i-p_i^{S})+\Big(\sum_{k\in T:k> i}\alpha_k\Big)(p_i-p_i^{S}) \\
\geq & \Big(\sum_{k\in S:k\geq i}\alpha_k-\beta_i\Big)(p_i-p_i^{S}) && (6.64)\\
& +\Big(\sum_{m\in T_r^*}\alpha_m\Big)(p_i-p_i^{S}), && (6.65)
\end{aligned}
$$

where the equality holds because $p_i^{S\cup T} = p_i^S$ and the inequality because $T_r^* \subseteq \{k \in T : k > i\}$. Observe that expression (6.59) exceeds expression (6.56), since $p_i^T = p_i$. Furthermore, the sum of expressions (6.58) and (6.64) exceeds expression (6.54). Therefore, showing that $v(S) + v(T) \leq v(S \cup T)$ boils down to showing that the sum of expressions (6.61) and (6.65) exceeds the sum of expressions (6.55) and (6.57). Note that, because $|S \cap T| = 1$, we have the following lower bound for (6.61):

$$\begin{aligned}
&\sum_{U\in\mathcal{C}(S\cup T)} \sum_{l,m\in U:l<m} (\alpha_m p_l^{S\cup T} - \alpha_l p_m^{S\cup T})_+ \\
\geq &\sum_{U\in\mathcal{C}(S)} \sum_{l,m\in U:l<m} (\alpha_m p_l^{S\cup T} - \alpha_l p_m^{S\cup T})_+ &(6.66)\\
&+ \sum_{U\in\mathcal{C}(T)} \sum_{l,m\in U:l<m} (\alpha_m p_l^{S\cup T} - \alpha_l p_m^{S\cup T})_+. &(6.67)
\end{aligned}$$

Obviously, (6.66) coincides with (6.55), since $p_l^{S\cup T} = p_l^S$ for all $l \in S$. So we only need to show that the sum of expressions (6.67) and (6.65) exceeds expression (6.57). For (6.67) we have that

$$\begin{aligned}
&\sum_{U\in\mathcal{C}(T)} \sum_{l,m\in U:l<m} (\alpha_m p_l^{S\cup T} - \alpha_l p_m^{S\cup T})_+ \\
= &\sum_{U\in\mathcal{C}(T)} \sum_{l,m\in U\backslash\{i\}:l<m} (\alpha_m p_l^{S\cup T} - \alpha_l p_m^{S\cup T})_+ + \sum_{l\in T_l^*} (\alpha_i p_l^{S\cup T} - \alpha_l p_i^{S\cup T})_+ \\
&+ \sum_{m\in T_r^*} (\alpha_m p_i^{S\cup T} - \alpha_i p_m^{S\cup T})_+ \\
\geq &\sum_{U\in\mathcal{C}(T)} \sum_{l,m\in U\backslash\{i\}:l<m} (\alpha_m p_l^{T} - \alpha_l p_m^{T})_+ + \sum_{l\in T_l^*} (\alpha_i p_l^{T} - \alpha_l p_i^{T})_+ &(6.68)\\
&+ \sum_{m\in T_r^*} (\alpha_m p_i^{S} - \alpha_i p_m^{T})_+. &(6.69)
\end{aligned}$$

The inequality holds since $p_j^{S\cup T} = p_j^T$ for all $j \in T\backslash\{i\}$, $p_i^{S\cup T} \leq p_i^T$ and $p_i^{S\cup T} = p_i^S$. Now adding (6.65) and (6.69) yields

$$\left(\sum_{m\in T_r^*} \alpha_m\right)(p_i - p_i^S) + \sum_{m\in T_r^*} (\alpha_m p_i^S - \alpha_i p_m^T)_+$$

$$\begin{aligned} &\geq \sum_{m \in T_r^*} (\alpha_m p_i - \alpha_i p_m^T)_+ \\ &= \sum_{m \in T_r^*} (\alpha_m p_i^T - \alpha_i p_m^T)_+, \end{aligned} \tag{6.70}$$

where the inequality holds because of Lemma 6.3.4 by taking $a_1 = \alpha_i$, $a_2 = \alpha_m$, $q_1 = p_i$, $\bar{q}_1 = p_i^S$ and $q_2 = p_m^T$. The equality is satisfied because $p_i^T = p_i$. Observe that the sum of expressions (6.68) and (6.70) coincides with expression (6.57). □

We remark that superadditivity together with Lemma 6.3.6 implies that 3-player cps games are convex. As promised, we now show that if $\sigma \in \Pi(N)$ is permutationally convex, then σ_1 is permutationally convex as well.

Theorem 6.3.3 Let $(N, \sigma_0, \alpha, \beta, p, \bar{p})$ be a cps situation and (N, v) its corresponding cps game. If $\sigma \in \Pi(N)$ is permutationally convex for (N, v), then σ_1 is permutationally convex for (N, v) as well. In particular, $m^{\sigma_1}(v) \in C(v)$.

Proof: We need to show for all $i, k \in \{0, \ldots, |N| - 1\}$ with $i < k$, and all $S \subseteq N \backslash [\sigma_1(k), \sigma_1]$ with $S \neq \emptyset$ that

$$v([\sigma_1(i), \sigma_1] \cup S) + v([\sigma_1(k), \sigma_1]) \leq v([\sigma_1(k), \sigma_1] \cup S) + v([\sigma_1(i), \sigma_1]).$$

If $i = 0$, then the inequality is trivially satisfied because (N, v) is superadditive. If $i \geq 2$, then the inequality is satisfied since $[\sigma_1(i), \sigma_1] = [\sigma(i), \sigma]$, $[\sigma_1(k), \sigma_1] = [\sigma(k), \sigma]$ and because of our assumption that σ is permutationally convex. So let $i = 1$. Since $([\sigma_1(i), \sigma_1] \cup S) \cap [\sigma_1(k), \sigma_1] = \{\sigma_1(1)\}$, the inequality is satisfied by Lemma 6.3.6 and the fact that $v(\{j\}) = 0$ for each $j \in N$. □

The final theorem of this section shows a way to alter a core element slightly, such that the new allocation is still in the core.

Theorem 6.3.4 Let $(N, \sigma_0, \alpha, \beta, p, \bar{p})$ be a cps situation and (N, v) its corresponding cps game. Let $z \in C(v)$ and let $j, k \in N$ with $j \neq k$. Furthermore let $\lambda \geq 0$ be such that $\lambda \leq (v(\{j, k\}) - z_k)_+$ and let $\bar{z}$ be such that $\bar{z}_i = z_i$ for all $i \in N \backslash \{j, k\}$, $\bar{z}_j = z_j - \lambda$ and $\bar{z}_k = z_k + \lambda$. Then, $\bar{z} \in C(v)$.

Proof: If $\lambda = 0$, then $\bar{z} = z$. Trivially $\bar{z} \in C(v)$. So assume $\lambda > 0$. It follows by definition of λ that $v(\{j,k\}) - z_k > 0$ and that $z_k + \lambda \leq v(\{j,k\})$.

Showing that $\bar{z} \in C(v)$ boils down to showing that for each $S \subseteq N \backslash \{k\}$ with $j \in S$, $\sum_{i \in S} \bar{z}_i \geq v(S)$. Let $S \subseteq N \backslash \{k\}$ be such that $j \in S$. Now note

$$\sum_{i \in S \cup \{k\}} z_i \geq v(S \cup \{k\}) \geq v(S) + v(\{j,k\}), \tag{6.71}$$

where the first inequality holds because $z \in C(v)$ and the second by Lemma 6.3.6. Thus

$$\sum_{i \in S} \bar{z}_i = \sum_{i \in S} z_i - \lambda = \sum_{i \in S \cup \{k\}} z_i - z_k - \lambda \geq v(S) + v(\{j,k\}) - z_k - \lambda \geq v(S),$$

where the first inequality follows by expression (6.71), and the second because $z_k + \lambda \leq v(\{j,k\})$. $\square$

Theorem 6.3.4 enables us to show that an even number of marginal vectors is in the core of a cps game. Let $(N, \sigma_0, \alpha, \beta, p, \bar{p})$ be a cps situation and (N, v) its corresponding cps game. Let $\sigma \in \Pi(N)$ be such that $m^\sigma(v) \in C(v)$. If k and j are the first and second player with respect to σ, respectively, then $m_k^\sigma(v) = 0$ and $m_j^\sigma(v) = v(\{j,k\})$. Let $\lambda = v(\{j,k\})$. According to Theorem 6.3.4, $\bar{z} \in C(v)$, with $\bar{z}$ given by $\bar{z}_k = m_k^\sigma(v) + \lambda = v(\{j,k\})$, $\bar{z}_j = m_j^\sigma(v) - \lambda = 0$ and $\bar{z}_i = m_i^\sigma(v)$ for all $i \in N \backslash \{j,k\}$. Observe that $\bar{z} = m^{\sigma_1}(v)$, and thus that $m^{\sigma_1}(v) \in C(v)$. Therefore we have the following proposition.

Proposition 6.3.1 Let $(N, \sigma_0, \alpha, \beta, p, \bar{p})$ be a cps situation and (N, v) its corresponding cps game. If $\sigma \in \Pi(N)$ is such that $m^\sigma(v) \in C(v)$, then $m^{\sigma_1}(v) \in C(v)$. In particular, the number of marginal vectors in $C(v)$ is even.

6.3.3 Convexity

In this section we investigate convexity of cps games. We will show that cps situations with equal completion time cost coefficients, equal crash time cost coefficients and equal maximal crash times lead to convex cps games.

Furthermore we show that by relaxing these conditions, convexity might be lost.

Theorem 6.3.5 Let $(N, \sigma_0, \alpha, \beta, p, \bar{p})$ be a cps situation with $\alpha_i = \alpha_j$, $\beta_i = \beta_j$ and $p_i - \bar{p}_i = p_j - \bar{p}_j$ for all $i, j \in N$, and let (N, v) be its corresponding cps game. Then (N, v) is convex.

Proof: For notational convenience we write $\alpha_i = \alpha$, $\beta_i = \beta$ and $p_i - \bar{p}_i = q$ for all $i \in N$, where α, β and q denote scalars and not vectors. We will first show that (N, v) is equal to the sum of the corresponding standard sequencing game and a symmetric game. We then show that this symmetric game is convex. Since standard sequencing games are convex as well, it follows that (N, v) is convex.

Let $S \subseteq N$. Each optimal processing schedule for S can be reached by first interchanging the jobs, and then reducing the initial processing times to the optimal processing times. We claim that the cost savings due to reducing the initial processing times can be easily determined. This can be seen as follows. Let $\sigma \in A(S)$ be a processing order, and let $k \in S$ be the l-th job of S with respect to σ. Then there are $|S| - l$ jobs of S in the queue behind k. Hence, if job k crashes, then this yields cost savings of $((|S| - l + 1)\alpha - \beta)q$. This term is non-negative only if $(|S| - l + 1)\alpha - \beta \geq 0$. Observe that these cost savings do not depend on σ or on k. We conclude that the cost savings for coalition S, due to crashing jobs will equal $q \sum_{l=1}^{|S|} ((|S| - l + 1)\alpha - \beta)_+$. Note that this expression only depends on the size of S, and not on S itself.

Because the cost savings obtained from optimally crashing jobs are independent of the order in which the jobs are processed, it follows that the total cost savings are maximised if the cost savings obtained from interchanging jobs are maximised. In particular, the total cost savings are maximised if the jobs are lined up in order of decreasing urgencies. Therefore

$$v(S) = \sum_{T \in \mathcal{C}(S)} \sum_{i,j \in T : i < j} (\alpha_j p_i - \alpha_i p_j)_+ + q \sum_{l=1}^{|S|} ((|S| - l + 1)\alpha - \beta)_+.$$

In Curiel, Pederzoli, and Tijs (1989) it is shown that (N, z), with $z(S) = \sum_{T \in \mathcal{C}(S)} \sum_{i,j \in T : i < j} (\alpha_j p_i - \alpha_i p_j)_+$ for each $S \subseteq N$, is convex. Hence, for convexity of (N, v) it is sufficient to show that (N, w) is convex, where $w(S) = q \sum_{l=1}^{|S|}((|S| - l + 1)\alpha - \beta)_+$ for each $S \subseteq N$. So let $i, j \in N$, $i \neq j$, and $S \subseteq N \backslash \{i, j\}$. We distinguish between three cases in order to show $w(S \cup \{i\}) + w(S \cup \{j\}) \leq w(S \cup \{i, j\}) + w(S)$.

`Case 1:` $w(S \cup \{i\}) = w(S \cup \{j\}) = 0$.
Trivially, $w(S \cup \{i\}) + w(S \cup \{j\}) = 0 \leq w(S \cup \{i, j\}) + w(S)$.

`Case 2:` $w(S \cup \{i\}) = w(S \cup \{j\}) > 0$ and $w(S) = 0$.
Since $w(S \cup \{i\}) > 0$ it follows that $q > 0$. Because $w(S) = 0$ and $q > 0$, $|S|\alpha - \beta \leq 0$. Therefore, $w(S \cup \{i\}) = w(S \cup \{j\}) = ((|S| + 1)\alpha - \beta)q$. Hence,

$$\begin{aligned} w(S \cup \{i\}) + w(S \cup \{j\}) &= 2((|S| + 1)\alpha - \beta)q \\ &\leq ((|S| + 1)\alpha - \beta)q + ((|S| + 2)\alpha - \beta)q \\ &= w(S \cup \{i, j\}) \\ &= w(S \cup \{i, j\}) + w(S). \end{aligned}$$

`Case 3:` $w(S) > 0$.
Because $w(S) > 0$, $w(S \cup \{i\}) = ((|S| + 1)\alpha - \beta)q + w(S)$. Furthermore, $w(S \cup \{i, j\}) = ((|S| + 2)\alpha - \beta)q + w(S \cup \{j\})$. Therefore,

$$\begin{aligned} w(S \cup \{i\}) + w(S \cup \{j\}) &= ((|S| + 1)\alpha - \beta)q + w(S) + w(S \cup \{j\}) \\ &\leq ((|S| + 2)\alpha - \beta)q + w(S) + w(S \cup \{j\}) \\ &= w(S \cup \{i, j\}) + w(S). \qquad \square \end{aligned}$$

Let $(N, \sigma_0, \alpha, \beta, p, \bar{p})$ be a cps situation with $\alpha_i = \alpha_j$, $\beta_i = \beta_j$ and $p_i - \bar{p}_i = p_j - \bar{p}_j$ for all $i, j \in N$. Let (N, v) be its corresponding cps game. According to the proof of Theorem 6.3.5, (N, v) is the sum of a symmetric game and the standard sequencing game. Hence, the Shapley value of (N, v) coincides with the sum of the Shapley value of the symmetric game and the Shapley value of the standard sequencing game. Since both can be computed easily it follows that the Shapley value of cps games arising from these special cps situations can be computed easily.

The following examples show that by relaxing the conditions of Theorem 6.3.5 convexity might be lost. The first example shows that cps games, arising from cps situations with equal completion time cost coefficients, equal crash time cost coefficients and equal crashed processing times, need not be convex.

Example 6.3.2 Let $(N, \sigma_0, \alpha, \beta, p, \bar{p})$ be given by $N = \{1,2,3,4\}$, $\alpha_i = 2$, $\beta_i = 5$ and $\bar{p}_i = 2$ for all $i \in N$. Furthermore, let $p = (7,3,3,7)$. Let (N, v) be its corresponding cps game. For each coalition, the optimal processing schedule can be reached by first interchanging jobs, and then crashing them. Since $2\alpha < \beta$, it follows that 2-player coalitions cannot obtain positive cost savings by crashing jobs. Since coalition $\{1,3\}$ is not allowed to rearrange the processing order of its jobs, we conclude that $v(\{1,3\}) = 0$. Because $2\alpha < \beta$ and $3\alpha > \beta$, coalition N will decide to crash exactly two jobs. In particular, the jobs in the first and second position of the optimal processing order will be crashed. Using this fact, it is straightforward to see that the schedule $((2,3,1,4),(7,2,2,7)) \in AS(N)$ is optimal. As a result, $v(N) = 20$. Now observe that $((2,3,1,4),(7,2,3,7)) \in AS(\{1,2,3\})$ and that $((1,2,3,4),(2,3,3,7)) \in AS(\{1,3,4\})$. These schedules yield cost savings of 17 and 5 for coalitions $\{1,2,3\}$ and $\{1,3,4\}$, respectively. Therefore $v(\{1,2,3\}) \geq 17$ and $v(\{1,3,4\}) \geq 5$. We conclude that $v(\{1,2,3\}) + v(\{1,3,4\}) \geq 22 > 20 = v(N) + v(\{1,3\})$, and thus that (N, v) is not convex. $\diamond$

The following example shows that cps games, arising from cps situations with equal completion time cost coefficients, equal processing times, and equal crashed processing times, need not be convex.

Example 6.3.3 Let $(N, \sigma_0, \alpha, \beta, p, \bar{p})$ be given by $N = \{1,2,3,4,5\}$, $\alpha_i = 2$, $p_i = 2$ and $\bar{p}_i = 1$ for all $i \in N$. Furthermore, let $\beta = (6,6,3,3,6)$. Let (N, v) be its corresponding cps game. For each coalition, the optimal processing schedule can be reached by first interchanging jobs, and then crashing them. Since $\alpha_i = 2$ and $p_i = 2$ for all $i \in N$, it follows that $\alpha_i p_j - \alpha_j p_i = 0$ for all $i, j \in N$. That is, first rearranging jobs yields no cost savings and no extra costs. Hence, the cost savings for each coalition consist of cost savings due to crashing only. Because $\alpha_i = 2$ and $p_i - \bar{p}_i = 1$ for all $i \in N$, it is optimal to put the jobs with lowest β_i to the front of the queue as much as possible. In particular, for coalitions $\{1,3,4\}$ and N the optimal schedules are $((1,2,3,4,5),(2,2,1,2,2)) \in AS(\{1,3,4\})$ and $((3,4,1,2,5),(2,2,1,1,2)) \in AS(N)$, respectively. This yields cost savings of $v(\{1,3,4\}) = 1$ and $v(N) = 12$. For coalitions $\{1,2,3,4\}$ and $\{1,3,4,5\}$ the optimal schedules are given by $((3,4,1,2,5),(2,2,1,1,2)) \in AS(\{1,2,3,4\})$ and $((1,2,3,4,5),(1,2,1,1,2)) \in AS(\{1,3,4,5\})$, respectively, with cost savings $v(\{1,2,3,4\}) = 8$ and $v(\{1,3,4,5\}) = 6$. We conclude that $v(\{1,2,3,4\}) + v(\{1,3,4,5\}) = 14 > 13 = v(N) + v(\{1,3,4\})$, and thus that (N,v) is not convex. $\diamond$

The last example shows that cps games, arising from cps situations with equal crash time cost coefficients, equal processing times, and equal crashed processing times, need not be convex.

Example 6.3.4 Let $(N,\sigma_0,\alpha,\beta,p,\bar{p})$ be given by $N = \{1,2,3,4\}$, $\beta_i = 5$, $p_i = 2$ and $\bar{p}_i = 1$ for all $i \in N$. Furthermore, let $\alpha = (1,1,5,1)$. Let (N,v) be its corresponding cps game. Since coalition $\{1,3\}$ is a disconnected coalition, it can only obtain cost savings by crashing job 1. Hence, $v(\{1,3\}) = 1$. For the grand coalition the schedule $((3,1,2,4),(2,2,1,2)) \in AS(N)$ is optimal, with cost savings $v(N) = 19$. Furthermore, $((3,1,2,4),(2,2,1,2)) \in AS(\{1,2,3\})$ and $((1,2,3,4),(1,2,1,2)) \in AS(\{1,3,4\})$. These schedules lead to cost savings of 18 and 3 for coalitions $\{1,2,3\}$ and $\{1,3,4\}$, respectively. We conclude that $v(\{1,2,3\}) + v(\{1,3,4\}) \geq 21 > 20 = v(N) + v(\{1,3\})$, and thus that (N,v) is not convex. $\diamond$

As a final remark we conjecture that if a cps situation $(N,\sigma_0,\alpha,\beta,p,\bar{p})$ satisfies $\alpha_i = \alpha_j$, $\beta_i = \beta_j$ and $p_i = p_j$ for all $i,j \in N$, then its corresponding cps game is convex.

6.4 Precedence sequencing games

In this section we study convexity of sequencing games arising from sequencing situations where precedence relations are imposed on the jobs. Precedence relations prescribe an order in which jobs have to be processed. Specifically, some jobs can only be processed if some other job(s) have been processed already. In practice many examples can be found where precedence relations play a role. For example, scheduling programs on a computer. In many cases one program needs the output of another program as input data. Another situation where precedence relations are involved is in the manufacturing of a car. Before you can paint the car you need to have the chassis, before you can place the wheels you need already the axles, etc.

In this section, which is based on Hamers, Klijn, and Van Velzen (2005), we are specifically interested in sequencing situations with chain precedence relations, and where the initial order is a concatenation of chains. A chain precedence relation is a precedence relation where each job is preceded by, and precedes, at most one job. A concatenation of chains arises if, for example, all jobs of a chain arrive at the machine at the same moment, and the jobs are initially ordered according to a first come first serve principle. We will show that sequencing games arising from these situations, i.e. sequencing situations with chain precedences and where the initial order is a concatenation of chains, are convex.

6.4.1 Precedence sequencing situations and games

In this section we first introduce precedence relations and subsequently we introduce precedence sequencing games.

A *precedence relation* $\mathcal{P}$ on a finite set N is a set of ordered pairs of N. A precedence relation is called feasible if it does not contain circuits. Formally, $\mathcal{P}$ is feasible if $(i,i) \notin \mathcal{P}$ for each $i \in N$, and if there are no $j_1, \dots, j_k \in N$ with $(j_m, j_{m+1}) \in \mathcal{P}$ for each $m \in \{1, \dots, k-1\}$, and $(j_k, j_1) \in \mathcal{P}$. We assume throughout this section that precedence relations are feasible. Furthermore we assume that precedence relations are minimal in the sense that there are no superfluous pairs in $\mathcal{P}$. More precisely, if $j_1, \dots, j_k \in N$ are such that

$(j_i, j_{i+1}) \in \mathcal{P}$ for each $i \in \{1, \ldots, k-1\}$, then $(j_1, j_k) \notin \mathcal{P}$.

A *precedence sequencing situation* is a tuple $(N, \mathcal{P}, \sigma_0, \alpha, p)$, where N, σ_0, α and p have the same interpretation as in Section 6.2. Additionally, there is a precedence relation $\mathcal{P}$ on N with the interpretation that if $(i, j) \in \mathcal{P}$, then the job of agent i has to be processed before the job of agent j. A processing order is called *feasible with respect to* $\mathcal{P}$ if for all $(i, j) \in \mathcal{P}$, i is processed before j. We denote the set of all feasible processing orders with respect to $\mathcal{P}$ by $Pr(N, \mathcal{P})$. Of course, we assume that $\sigma_0 \in Pr(N, \mathcal{P})$. A processing order is called *optimal* if it minimises the total cost of all agents. Formally, $\sigma \in Pr(N, \mathcal{P})$ is optimal if

$$\sum_{i \in N} C_i(\sigma) \leq \sum_{i \in N} C_i(\tau) \text{ for each } \tau \in Pr(N, \mathcal{P}).$$

In the remainder of this section we introduce precedence sequencing games. The worth of a coalition in a precedence sequencing game is the maximal cost savings obtainable by this coalition by means of an admissible and feasible rearrangement of the initial order. Formally, $\sigma \in Pr(N, \mathcal{P})$ is called admissible for coalition $S \subseteq N$ if it satisfies (6.1). The set of admissible and feasible processing orders for coalition $S \subseteq N$ is denoted by $AF(S)$. Given a precedence sequencing situation $(N, \mathcal{P}, \sigma_0, \alpha, p,)$ the corresponding *precedence sequencing game* (N, v) is defined by

$$v(S) = \sum_{i \in S} C_i(\sigma_0) - \min_{\sigma \in AF(S)} \sum_{i \in S} C_i(\sigma)$$

for each $S \subseteq N$. It is straightforward to see that precedence sequencing games are chain-component additive games with respect to σ_0. So cores of precedence sequencing games are non-empty.

The following example illustrates a precedence sequencing game in case the directed graph associated to the precedence relation is a tree.

Example 6.4.1 Let $(N, \mathcal{P}, \sigma_0, \alpha, p)$ be a precedence sequencing situation, with $N = \{1, 2, 3, 4\}$, $\mathcal{P} = \{(1, 2), (2, 4), (1, 3)\}$, $\sigma_0 = (1, 2, 3, 4)$, $\alpha = (1, 2, 3, 4)$ and $p = (1, 1, 1, 1)$. Consider coalition $\{2, 3, 4\}$. The admissible and feasible processing orders for this coalition are $AF(\{2, 3, 4\}) =$

$\{(1,2,3,4),(1,3,2,4),(1,2,4,3)\}$. Both $(1,3,2,4)$ and $(1,2,4,3)$ yield cost savings of 1 for $\{2,3,4\}$. Hence, $v(\{2,3,4\}) = 1$.

It is straightforwardly verified that the worths of the other connected coalitions are given by $v(\{i\}) = 0$ for $i \in \{1,2,3,4\}$, $v(\{1,2\}) = 0$, and $v(S) = 1$ if $S = \{2,3\},\{3,4\},\{1,2,3\},\{1,2,3,4\}$. $\diamond$

6.4.2 Convexity

In this section we will establish convexity of precedence sequencing games corresponding to situations where the precedence relations consist of parallel chains and the initial orders are concatenations of these chains.

The following example shows that a precedence sequencing game arising from a sequencing situation where the directed graph associated to the precedence relation is a tree need not be convex.

Example 6.4.2 Consider the precedence sequencing game of Example 6.4.1. Then $v(\{2,3\}) + v(\{3,4\}) = 2 > 1 = v(\{2,3,4\}) + v(\{3\})$, which implies that (N,v) is not convex. $\diamond$

Let $(N,\mathcal{P},\sigma_0,\alpha,p)$ be a precedence sequencing situation. Then $\mathcal{P}$ is said to be a *network of parallel chains* if each player precedes at most one player, and is preceded by at most one player. Formally, if for each $i \in N$, $|\{j \in N : (i,j) \in \mathcal{P}\}| \leq 1$ and $|\{j \in N : (j,i) \in \mathcal{P}\}| \leq 1$. A chain is a maximal ordered set of players $\{i_1,\ldots,i_k\}$ with $(i_l,i_{l+1}) \in \mathcal{P}$ for each $l \in \{1,\ldots,k-1\}$.

Let $(N,\mathcal{P},\sigma_0,\alpha,p)$ be a precedence sequencing situation where $\mathcal{P}$ is a network of parallel chains. Let $P(c)$, $c \in \{1,\ldots,C\}$, denote these chains. Obviously, the sets $P(c)$, $c \in \{1,\ldots,C\}$, form a partition of N. We will show that if σ_0 is a concatenation of these chains, then the corresponding precedence sequencing game is convex. First we illustrate concatenations of chains.

Example 6.4.3 Let $(N,\mathcal{P},\sigma_0,\alpha,p)$ be a precedence sequencing situation, where $N = \{1,2,3,4,5,6\}$, $\mathcal{P} = \{(1,2),(3,4),(4,5),(5,6)\}$, $\alpha = (2,5,6,6,3,6)$ and $p_i = 1$ for each $i \in N$. The precedence relation $\mathcal{P}$ constitutes two chains, $\{1,2\}$ and $\{3,4,5,6\}$. If σ_0 is a concatenation of chains, then either $\sigma_0 = (1,2,3,4,5,6)$ or $\sigma_0 = (3,4,5,6,1,2)$. $\diamond$

Let $(N, \mathcal{P}, \sigma_0, \alpha, p)$ be a precedence sequencing situation where $\mathcal{P}$ is a network of parallel chains, and σ_0 a concatenation of these chains. Let $P(c)$, $c \in \{1, \ldots, C\}$, denote these chains. Without loss of generality we assume, throughout the remainder of this section, that the first positions of σ_0 are taken by the members of $P(1)$, followed by the members of $P(2)$, etc. Let $S \subseteq N$ be a connected coalition. Then there are $c^*, d^* \in \{1, \ldots, C\}$, $c^* \leq d^*$, with

1. $S \cap P(c) = \emptyset$ for each $c \in \{1, \ldots, C\}$ with $c < c^*$ or $c > d^*$;
2. $S \cap P(c) \neq \emptyset$ for $c = c^*$ and $c = d^*$;
3. $P(c) \subseteq S$ for each $c \in \{1, \ldots, C\}$ with $c^* < c < d^*$.

For any $c \in \{c^*, \ldots, d^*\}$, let $ch_c(S) = S \cap P(c)$ be the (non-empty) intersection of S with the players of chain c. Observe that each $ch_c(S)$ inherits in a natural way the ordering induced by σ_0, and note that $ch_c(S) = P(c)$ for each $c \in \{1, \ldots, C\}$ with $c^* < c < d^*$.

Before stating Sidney's algorithm, that provides a method to calculate an optimal order for each connected coalition in case the precedence relation is a network of parallel chains, we introduce a few more notations and definitions. Let $U = \{i_1, \ldots, i_k\}$ be a connected subset of some chain $P(c)$, such that $(i_l, i_{l+1}) \in \mathcal{P}$ for each $l \in \{1, \ldots, k-1\}$. A *head* of U is a set $T \subseteq U$ such that $T = \{i_1, \ldots, i_l\}$ for some $l \in \{1, \ldots, k\}$. Similarly, a *tail* of U is a set $T \subseteq U$ such that $T = \{i_l, \ldots, i_k\}$ for some $l \in \{1, \ldots, k\}$. For any $T \subseteq N$, $T \neq \emptyset$, we define $\alpha(T) = \sum_{i \in T} \alpha_i$, $p(T) = \sum_{i \in T} p_i$ and $u(T) = \frac{\alpha(T)}{p(T)}$, where $u(T)$ is called the urgency index of coalition T.

Sidney's algorithm: **Optimal order of connected S**

Step 1: Construction of Sidney-components

For every $c \in \{c^*, \ldots, d^*\}$, find the following coalitions:

T_1^c, the largest head of $ch_c(S)$ that satisfies

$$u(T_1^c) = \max\{u(T) : T \text{ is a head of } ch_c(S)\}.$$

For $m > 1$, let T_m^c be the largest head of $ch_c(S) \backslash \left(\bigcup_{i=1}^{m-1} T_i^c \right)$ that satisfies

$$u(T_m^c) = \max\{u(T) : T \text{ is a head of } ch_c(S) \backslash \left(\bigcup_{i=1}^{m-1} T_i^c \right)\}.$$

Let m_c be the number of sets we obtain in this way. Then, $\bigcup_{r=1}^{m_c} T_r^c = ch_c(S)$. The sets T_r^c ($c \in \{c^*, \ldots, d^*\}$ and $r \in \{1, \ldots, m_c\}$) are called the *Sidney-components* of S.

Step 2: Ordering Sidney-components

Order the Sidney-components of S in weakly decreasing order with respect to their urgency indices.

The following theorem follows from Sidney (1975).

Theorem 6.4.1 An order resulting from Sidney's algorithm is admissible, feasible and optimal for S.

Example 6.4.4 Let $(N, \mathcal{P}, \sigma_0, \alpha, p)$ be as defined in Example 6.4.3 with $\sigma_0 = (1, 2, 3, 4, 5, 6)$. Let $S = \{2, 3, 4, 5, 6\}$. Then $ch_1(S) = \{2\}$ and $ch_2(S) = \{3, 4, 5, 6\}$. Following the first step of Sidney's algorithm we obtain $T_1^1 = \{2\}$, $T_1^2 = \{3, 4\}$ and $T_2^2 = \{5, 6\}$, with $u(\{2\}) = 5$, $u(\{3, 4\}) = 6$ and $u(\{5, 6\}) = 4\frac{1}{2}$, respectively. From the second step of the algorithm and Theorem 6.4.1 it follows that processing the jobs in the order $(1, 3, 4, 2, 5, 6)$ is optimal for coalition S given the precedence relation $\mathcal{P}$. $\diamond$

The following lemmas, which describe relations between urgency indices, facilitate the proof of our main result.

Lemma 6.4.1 Let $S, T \subseteq N$ be disjoint and non-empty. If $u(S) \geq u(T)$, then $u(S) \geq u(S \cup T) \geq u(T)$. If $u(S) > u(T)$, then $u(S) > u(S \cup T) > u(T)$. If $u(S) = u(T)$, then $u(S) = u(S \cup T) = u(T)$.

Proof: Note that

$$u(S \cup T) = \frac{p(S)}{p(S)+p(T)} u(S) + \frac{p(T)}{p(S)+p(T)} u(T).$$

All assertions of the lemma follow directly from the observation that $u(S \cup T)$ is a convex combination of $u(S)$ and $u(T)$. □

Lemma 6.4.2 Let $S, T, W \subseteq N$ be pairwise disjoint and non-empty. If $u(W) \geq u(T) \geq u(S)$, then $u(S \cup T \cup W) \geq u(S \cup T)$.

Proof: Because $u(T) \geq u(S)$ it follows from Lemma 6.4.1 that $u(T) \geq u(S \cup T) \geq u(S)$, and therefore $u(W) \geq u(S \cup T)$. Applying Lemma 6.4.1 again gives $u(W) \geq u(S \cup T \cup W) \geq u(S \cup T)$. □

Lemma 6.4.3 Let $T \subseteq N$, $T \neq \emptyset$ and let $T_1^c, \ldots, T_{m_c}^c$ be the Sidney-components of T for some chain c. Then $u(T_1^c) > u(T_2^c) > \cdots > u(T_{m_c}^c)$.

Proof: Follows immediately from the definition of the Sidney-components and Lemma 6.4.1. □

To prove our main result we need the following notation. For two coalitions $U, V \subseteq N$ with $U \cap V = \emptyset$, we define

$$g(U, V) := \max\{0, \alpha(V)p(U) - \alpha(U)p(V)\}.$$

Observe that $g(U, V) \geq 0$, and that $g(U, V) > 0$ if and only if $u(V) > u(U)$. So $g(U, V)$ are the cost savings that can be obtained by interchanging the jobs of coalitions U and V, if the jobs of U are ordered directly in front of the jobs of V. Extending to two collections $\mathcal{U}, \mathcal{V} \subseteq 2^N$ with $U \cap V = \emptyset$ for each $U \in \mathcal{U}, V \in \mathcal{V}$, we define

$$G(\mathcal{U}, \mathcal{V}) := \sum_{U \in \mathcal{U}, V \in \mathcal{V}} g(U, V). \tag{6.72}$$

Before we prove convexity of precedence sequencing games, we recall a theorem of Borm, Fiestras-Janeiro, Hamers, Sánchez, and Voorneveld (2002). This theorem show that convexity of chain-component additive games is equivalent to a restricted set of inequalities.

Theorem 6.4.2 (Borm, Fiestras-Janeiro, Hamers, Sánchez, and Voorneveld (2002)) Let $\sigma_0 : \{1, \ldots, |N|\} \to N$ be an order on N, and let (N, v) be chain-component additive with respect to σ_0. Then (N, v) is convex if and only if

$$v(S \cup \{i\}) + v(S \cup \{j\}) \leq v(S \cup \{i, j\}) + v(S) \tag{6.73}$$

for each $i, j \in N$, $i \neq j$, and each connected $S \subseteq N \backslash \{i, j\}$ such that $S \cup \{i\}$ and $S \cup \{j\}$ are connected as well.

Let $\sigma_0 : \{1, \ldots, |N|\} \to N$ be an order on N and $i, j \in N$, $i \neq j$. Then the only connected coalition $S \subseteq N \backslash \{i, j\}$ such that $S \cup \{i\}$ and $S \cup \{j\}$ are connected as well, is the coalition consisting of the players "in between" i and j, with respect to σ_0. Now we are ready to state and prove our main theorem.

Theorem 6.4.3 Let $(N, \mathcal{P}, \sigma_0, \alpha, p)$ be a precedence sequencing situation where $\mathcal{P}$ is a network of parallel chains and σ_0 a concatenation of these chains. Then the corresponding precedence sequencing game (N, v) is convex.

Proof: According to Theorem 6.4.2 we need to show that (6.73) is satisfied for all $i, j \in N$, $i \neq j$, and $S \subseteq N \backslash \{i, j\}$ such that S consists of all players in between i and j, with respect to σ_0. So let $i, j \in N$, $i \neq j$, and $S \subseteq N \backslash \{i, j\}$ consist of all players in between i and j, with respect to σ_0.

If i and j are in the same chain, then no reordering of the players is feasible, and therefore $v(S \cup \{i, j\}) = v(S \cup \{j\}) = v(S \cup \{i\}) = v(S) = 0$. Obviously, (6.73) is satisfied in this case. So assume that i is an element of chain $P(c^*)$ and j is an element of chain $P(d^*)$, with $c^* < d^*$. We will now partition each of the coalitions S, $S \cup \{i\}$, $S \cup \{j\}$ and $S \cup \{i, j\}$ into four sets in order to obtain a usable expression for (6.73). In particular we use that some elements of the partition of S coincide with elements of the partitions of $S \cup \{i\}$ and $S \cup \{j\}$. Similarly, some elements of the partition of $S \cup \{i, j\}$ coincide with elements of the partitions of $S \cup \{i\}$ and $S \cup \{j\}$.

For an illustration of these sets we refer to Figure 6.1, where the Sidney-components are the connected sets of jobs that have the same color.[7]

For $V = S \cup \{i,j\}, S \cup \{i\}$ let $C_1(V)$ be the collection of Sidney-components of V that are contained in $P(c^*)$ and that are not Sidney-components of $S \cup \{j\}$. Note that $C_1(S \cup \{i,j\}) = C_1(S \cup \{i\})$, because $P(c^*) \cap (S \cup \{i,j\}) = P(c^*) \cap (S \cup \{i\})$. From Lemma 6.4.4, which is stated and proved in Section 6.4.3, it follows that $C_1(S \cup \{i,j\})$ contains only one Sidney-component. Let U^* be the unique element of $C_1(S \cup \{i,j\}) = C_1(S \cup \{i\})$.

For $V = S \cup \{j\}, S$ let $C_1(V)$ be the collection of Sidney-components of V that are contained in $P(c^*)$ and that are not Sidney-components of $S \cup \{i,j\}$. Note that $C_1(S \cup \{j\}) = C_1(S)$.

For $V = S \cup \{i,j\}, S \cup \{j\}$ let $C_4(V)$ be the collection of Sidney-components of V that are contained in $P(d^*)$ and that are not Sidney-components of $S \cup \{i\}$. Note that $C_4(S \cup \{i,j\}) = C_4(S \cup \{j\})$. From Lemma 6.4.4 it follows that $C_4(S \cup \{i,j\})$ contains only one Sidney-component. Let V^* be the unique element of $C_4(S \cup \{i,j\}) = C_4(S \cup \{j\})$.

For $V = S \cup \{i\}, S$ let $C_4(V)$ be the collection of Sidney-components of V that are contained in $P(d^*)$ and that are not Sidney-components of $S \cup \{i,j\}$. Note that $C_4(S \cup \{i\}) = C_4(S)$.

Note that, in case $C_1(S) = C_1(S \cup \{j\})$ is non-empty, then the last job of $C_1(V)$ with respect to σ_0 coincides in all four situations, i.e. $V = S \cup \{i,j\}, S \cup \{i\}, S \cup \{j\}, S$. This follows straightforwardly from Lemma 6.4.4. Similarly, in case $C_4(S) = C_4(S \cup \{i\}$ is non-empty, then the first job of $C_4(V)$ with respect to σ_0 coincides for $V = S \cup \{i,j\}, S \cup \{i\}, S \cup \{j\}, S$ as well.

[7] The collections of Sidney-components $C_2(.)$ and $C_3(.)$ are defined in Lemma 6.4.5, but can be ignored for the moment.

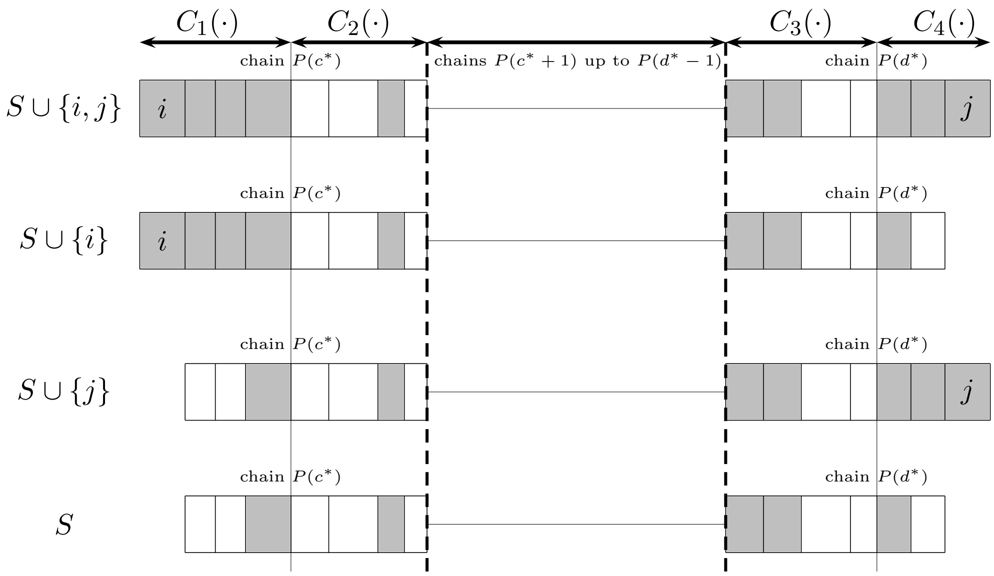

Figure 6.1: The sets $C_1(\cdot)$ up to $C_4(\cdot)$.

From Lemma 6.4.5 it follows that

$$\begin{aligned}
&v(S \cup \{i,j\}) - v(S \cup \{i\}) - v(S \cup \{j\}) + v(S) \\
&= G(C_1(S \cup \{i,j\}), C_4(S \cup \{i,j\})) - G(C_1(S \cup \{i\}), C_4(S \cup \{i\})) \\
&\quad - G(C_1(S \cup \{j\}), C_4(S \cup \{j\})) + G(C_1(S), C_4(S)) \\
&= G(\{U^*\}, \{V^*\}) - G(\{U^*\}, C_4(S \cup \{i\})) \\
&\quad - G(C_1(S \cup \{j\}), \{V^*\}) + G(C_1(S), C_4(S)) \\
&= g(U^*, V^*) - \sum_{V \in C_4(S \cup \{i\})} g(U^*, V) \\
&\quad - \sum_{U \in C_1(S \cup \{j\})} g(U, V^*) + \sum_{U \in C_1(S), V \in C_4(S)} g(U, V),
\end{aligned} \tag{6.74}$$

where the last equality holds by (6.72). Hence, (6.73) is satisfied if expression (6.74) is non-negative. We distinguish between 2 cases.

`Case 1:` $g(U^*, V^*) = 0$, i.e. $u(U^*) \geq u(V^*)$.

Because V^* is a Sidney-component, it follows from the definition of Sidney-components that $u(V^*) \geq u(V_1)$, where V_1 is the first Sidney-component in $C_4(S \cup \{i\})$.[8] Hence, $u(U^*) \geq u(V_1)$, and $g(U^*, V_1) = 0$. Using Lemma 6.4.3 it follows that $u(U^*) \geq u(V_1) \geq u(V)$ for each $V \in C_1(S \cup \{j\})$, and therefore that $\sum_{V \in C_4(S \cup \{i\})} g(U^*, V) = 0$. Similarly, it can be shown that $\sum_{U \in C_1(S \cup \{j\})} g(U, V^*) = 0$ and $\sum_{U \in C_1(S), V \in C_4(S)} g(U, V) = 0$. Therefore expression (6.74) is zero.

`Case 2:` $g(U^*, V^*) > 0$, i.e. $u(V^*) > u(U^*)$.

Define

$$\begin{aligned} V_a^* &= \bigcup_{V \in C_4(S \cup \{i\}) : g(U^*, V) > 0} V; \\ V_b^* &= V^* \backslash V_a^*. \end{aligned}$$

From Lemma 6.4.3 it follows that V_a^* is a head of V^* consisting of the players of the Sidney-components in $C_4(S \cup \{i\})$ with higher urgency index than U^*. Note that $j \in V_b^*$, and therefore $V_b^* \neq \emptyset$. Similarly we define

$$\begin{aligned} U_b^* &= \bigcup_{U \in C_1(S \cup \{j\}) : g(U, V^*) > 0} U; \\ U_a^* &= U^* \backslash U_b^*. \end{aligned}$$

From Lemma 6.4.3 it follows that U_b^* is a tail of U^* consisting of the players of the Sidney-components in $C_1(S \cup \{j\})$ with lower urgency index than V^*. Note that $i \in U_a^*$ and therefore $U_a^* \neq \emptyset$. Rewriting the first two terms of (6.74) we obtain

$$\begin{aligned} & g(U^*, V^*) - \sum_{V \in C_4(S \cup \{i\})} g(U^*, V) \\ = \;& g(U^*, V^*) - \sum_{V \in C_4(S \cup \{i\}) : V \subseteq V_a^*} g(U^*, V) \\ = \;& \alpha(V^*) p(U^*) - \alpha(U^*) p(V^*) \\ & - \sum_{V \in C_4(S \cup \{i\}) : V \subseteq V_a^*} \Big(\alpha(V) p(U^*) - \alpha(U^*) p(V) \Big) \\ = \;& \alpha(V^*) p(U^*) - \alpha(U^*) p(V^*) - \alpha(V_a^*) p(U^*) + \alpha(U^*) p(V_a^*) \\ = \;& \alpha(V_b^*) p(U^*) - \alpha(U^*) p(V_b^*), \end{aligned} \tag{6.75}$$

[8] If $C_4(S \cup \{i\}) = \emptyset$, then trivially $\sum_{V \in C_4(S \cup \{i\})} g(U^*, V) = 0$.

where the second equality follows from $u(V^*) > u(U^*)$ and $u(V) > u(U^*)$ for all $V \in C_4(S \cup \{i\})$ with $V \subseteq V_a^*$. Rewriting the last two terms of (6.74) we obtain

$$\begin{aligned}
& -\sum_{U \in C_1(S \cup \{j\})} g(U, V^*) + \sum_{U \in C_1(S), V \in C_4(S)} g(U, V) \\
\geq\ & -\sum_{U \in C_1(S \cup \{j\}): U \subseteq U_b^*} g(U, V^*) + \sum_{U \in C_1(S), V \in C_4(S): U \subseteq U_b^*, V \subseteq V_a^*} g(U, V) \\
\geq\ & -\sum_{U \in C_1(S \cup \{j\}): U \subseteq U_b^*} \Big(\alpha(V^*) p(U) - \alpha(U) p(V^*) \Big) \\
& + \sum_{U \in C_1(S), V \in C_4(S): U \subseteq U_b^*, V \subseteq V_a^*} \Big(\alpha(V) p(U) - \alpha(U) p(V) \Big) \\
=\ & -\alpha(V^*) p(U_b^*) + \alpha(U_b^*) p(V^*) + \alpha(V_a^*) p(U_b^*) - \alpha(U_b^*) p(V_a^*) \\
=\ & -\alpha(V_b^*) p(U_b^*) + \alpha(U_b^*) p(V_b^*). \qquad (6.76)
\end{aligned}$$

The first inequality follows from the definition of U_b^* and $g(U, V) \geq 0$ for all $U, V \subseteq N$. The second inequality follows from the definition of U_b^* and $g(U, V) \geq \alpha(V)p(U) - \alpha(U)p(V)$ for all $U, V \subseteq N$.

Substituting (6.75) and (6.76) in (6.74) we obtain

$$\begin{aligned}
& v(S \cup \{i, j\}) - v(S \cup \{i\}) - v(S \cup \{j\}) + v(S) \\
\geq\ & \alpha(V_b^*) p(U_a^*) - \alpha(U_a^*) p(V_b^*). \qquad (6.77)
\end{aligned}$$

To show that expression (6.77) is non-negative, we will prove that $u(V_b^*) \geq u(V^*)$ and $u(U^*) \geq u(U_a^*)$. This implies, using the assumption $u(V^*) > u(U^*)$, that $u(V_b^*) > u(U_a^*)$. As a result expression (6.77) is non-negative.

Suppose that $V_a^* = \emptyset$, then $V_b^* = V^*$ and hence $u(V_b^*) = u(V^*)$. So suppose that $V_a^* \neq \emptyset$ and suppose that $u(V_a^*) > u(V_b^*)$. Then, using Lemma 6.4.1 it follows that $u(V_a^*) > u(V^*) > u(V_b^*)$. This implies that V^* is not a Sidney-component of $S \cup \{i, j\}$, which is a contradiction. Hence, $u(V_b^*) \geq u(V_a^*)$ and using Lemma 6.4.1 it follows that $u(V_b^*) \geq u(V^*)$. The proof that $u(U^*) \geq u(U_a^*)$ runs similarly. □

Finally we illustrate that convexity might be lost if the initial order is not a concatenation of chains.

Example 6.4.5 Let the precedence sequencing situation $(N, \mathcal{P}, \sigma_0, \alpha, p)$ be given by $N = \{1,2,3\}$, $\mathcal{P} = \{(1,3)\}$, $\sigma_0 = (1,2,3)$, $\alpha = (1,2,3)$, and $p = (1,1,1)$. Hence, σ_0 is not a concatenation of chains. Let (N, v) be the corresponding precedence sequencing game. We leave it to the reader to verify that $v(\{2\}) = 0$, $v(\{1,2\}) = 1$, $v(\{2,3\}) = 1$ and $v(\{1,2,3\}) = 1$. Hence, $v(\{1,2\}) + v(\{2,3\}) = 2 > 1 = v(\{1,2,3\}) + v(\{2\})$. So (N, v) is not convex. $\diamond$

6.4.3 Proofs of lemmas

In this section we state and prove two lemmas needed for the proof of Theorem 6.4.3. We will use the notation introduced in the proof of Theorem 6.4.3.

Lemma 6.4.4 The sets $C_1(S \cup \{i,j\})$ and $C_4(S \cup \{i,j\})$ contain precisely one element (i.e. Sidney-component).

Proof: From Step 1 of Sidney's algorithm, and from the definition of $C_4(S \cup \{i,j\})$, it follows immediately that $C_4(S \cup \{i,j\})$ contains precisely one element.

We will now show that $C_1(S \cup \{i,j\})$ contains a single element. If i is the only player in $P(c^*) \cap (S \cup \{i,j\})$, then $C_1(S \cup \{i,j\}) = \{\{i\}\}$ and we are done. So assume that i is not the only player in $P(c^*) \cap (S \cup \{i,j\})$.

The Sidney-component of $S \cup \{i,j\}$ containing i is the union of $\{i\}$, a number of Sidney-components of $S \cup \{j\}$, and possibly a head of a Sidney-component of $S \cup \{j\}$. That is, the Sidney-component of $S \cup \{i,j\}$ containing i is of the form $\{i\} \cup \bigcup_{l=1}^{m-1} A_l \cup B$, where A_l is a Sidney-component of $S \cup \{j\}$ for each $l \in \{1, \ldots, m-1\}$ and B is a head of Sidney-component A_m. We show that $B = A_m$. Suppose to the contrary that $B \neq A_m$.

From Step 1 of Sidney's algorithm it follows that $u(\{i\} \cup \bigcup_{l=1}^{m-1} A_l \cup B) \geq u(\{i\} \cup \bigcup_{l=1}^{m-1} A_l)$. If $u(B) < u(\{i\} \cup \bigcup_{l=1}^{m-1} A_l)$, then it follows from Lemma 6.4.1 that $u(\{i\} \cup \bigcup_{l=1}^{m-1} A_l \cup B) < u(\{i\} \cup \bigcup_{l=1}^{m-1} A_l)$, which is a contradiction. Hence, $u(B) \geq u(\{i\} \cup \bigcup_{l=1}^{m-1} A_l)$.

Moreover, since A_m is a Sidney-component of $S \cup \{j\}$, $u(A_m \backslash B) \geq u(B)$. Hence, we have $u(A_m \backslash B) \geq u(B) \geq u(\{i\} \cup \bigcup_{l=1}^{m-1} A_l)$. From Lemma 6.4.2, by using $S = \{i\} \cup \bigcup_{l=1}^{m-1} A_l$, $T = B$ and $W = A_m \backslash B$, we obtain that $u(\{i\} \cup \bigcup_{l=1}^{m} A_l) \geq u(\{i\} \cup \bigcup_{l=1}^{m-1} A_l \cup B)$, which contradicts that the Sidney-component of $S \cup \{i,j\}$ containing i is $\{i\} \cup \bigcup_{l=1}^{m-1} A_l \cup B$. We conclude that $C_1(S \cup \{i,j\})$ contains a single element. □

Lemma 6.4.5 It holds that

$$\begin{aligned} & v(S \cup \{i,j\}) - v(S \cup \{i\}) - v(S \cup \{j\}) + v(S) \\ = \ & G(C_1(S \cup \{i,j\}), C_4(S \cup \{i,j\})) - G(C_1(S \cup \{i\}), C_4(S \cup \{i\})) \\ & -G(C_1(S \cup \{j\}), C_4(S \cup \{j\})) + G(C_1(S), C_4(S)). \end{aligned}$$

Proof: Besides the already introduced sets $C_1(V)$ and $C_4(V)$, where $V = S \cup \{i,j\}, S \cup \{i\}, S \cup \{j\}, S$, we introduce the following collections of Sidney-components (for an illustration see Figure 6.1). For $V = S \cup \{i,j\}, S \cup \{i\}$ let $C_2(V)$ be the collection of Sidney-components of V that are contained in $P(c^*)$ and that are also Sidney-components of $S \cup \{j\}$.

For $V = S \cup \{j\}, S$ let $C_2(V)$ be the collection of Sidney-components of V that are contained in $P(c^*)$ and that are also Sidney-components of $S \cup \{i,j\}$. Note that $C_2(S \cup \{i,j\}) = C_2(S \cup \{i\}) = C_2(S \cup \{j\}) = C_2(S)$.

For $V = S \cup \{i,j\}, S \cup \{j\}$ let $C_3(V)$ be the collection of Sidney-components of V that are contained in $P(d^*)$ and that are also Sidney-components of $S \cup \{i\}$.

For $V = S \cup \{i\}, S$ let $C_3(V)$ be the collection of Sidney-components of V that are contained in $P(d^*)$ and that are also Sidney-components of $S \cup \{i,j\}$. Note that $C_3(S \cup \{i,j\}) = C_3(S \cup \{i\}) = C_3(S \cup \{j\}) = C_3(S)$.

For $l \in \{c^*+1, \ldots, d^*-1\}$ let D_l be the collection of Sidney-components that are contained in chain $P(l)$.

Finally, for $V = S \cup \{i,j\}, S \cup \{i\}, S \cup \{j\}, S$ let $C_{12}(V) = C_1(V) \cup C_2(V)$ and let $C_{34}(V) = C_3(V) \cup C_4(V)$.

From Sidney's algorithm, the definition of the game (N,v), and (6.72) it follows that for $T = S \cup \{i,j\}, S \cup \{i\}, S \cup \{j\}, S$ we have

$$v(T) = G(C_{12}(T), C_{34}(T)) + \sum_{l=c^*+1}^{d^*-1} G(C_{12}(T), D_l) + \sum_{l,m\in\{c^*+1,\ldots,d^*-1\}:l<m} G(D_l, D_m) + \sum_{l=c^*+1}^{d^*-1} G(D_l, C_{34}(T)). \tag{6.78}$$

Now it follows that

$$\begin{aligned}
& v(S\cup\{i,j\}) - v(S\cup\{i\}) - v(S\cup\{j\}) + v(S) \\
= \; & G(C_{12}(S\cup\{i,j\}), C_{34}(S\cup\{i,j\})) - G(C_{12}(S\cup\{i\}), C_{34}(S\cup\{i\})) \\
& -G(C_{12}(S\cup\{j\}), C_{34}(S\cup\{j\})) + G(C_{12}(S), C_{34}(S)) \\
= \; & G(C_{1}(S\cup\{i,j\}), C_{34}(S\cup\{i,j\})) - G(C_{1}(S\cup\{i\}), C_{34}(S\cup\{i\})) \\
& -G(C_{1}(S\cup\{j\}), C_{34}(S\cup\{j\})) + G(C_{1}(S), C_{34}(S)) \\
= \; & G(C_{1}(S\cup\{i,j\}), C_{4}(S\cup\{i,j\})) - G(C_{1}(S\cup\{i\}), C_{4}(S\cup\{i\})) \\
& -G(C_{1}(S\cup\{j\}), C_{4}(S\cup\{j\})) + G(C_{1}(S), C_{4}(S))
\end{aligned}$$

The first equality follows from (6.78) and $C_{12}(S\cup\{i,j\}) = C_{12}(S\cup\{i\})$, $C_{12}(S\cup\{j\}) = C_{12}(S)$, $C_{34}(S\cup\{i,j\}) = C_{34}(S\cup\{j\})$, and $C_{34}(S\cup\{i\}) = C_{34}(S)$. The second equality follows from $C_k(S\cup\{i,j\}) = C_k(S\cup\{i\}) = C_k(S\cup\{j\}) = C_k(S)$ for $k = 2,3$, from $C_4(S\cup\{i,j\}) = C_4(S\cup\{j\})$ and from $C_4(S\cup\{i\}) = C_4(S)$. The third equality follows from $C_3(S\cup\{i,j\}) = C_3(S\cup\{i\}) = C_3(S\cup\{j\}) = C_3(S)$, from $C_1(S\cup\{i,j\}) = C_1(S\cup\{i\})$ and from $C_1(S\cup\{j\}) = C_1(S)$. This completes the proof of the lemma. □

6.5 Weak-relaxed sequencing games

In standard sequencing games, rearrangements of the initial order are admissible for a coalition if no jumps take place over agents outside this coalition. In Curiel, Potters, Rajendra Prasad, Tijs, and Veltman (1993) it is argued that this notion of admissibility is too restrictive. Specifically, agents outside a coalition will not object to jumps, as long as their completion times do not increase. Hence, it is suggested that a reordering should be admissible for a coalition, as long as the completion times of the agents outside the coalition do not increase. In this way the class of relaxed sequencing games is defined. It is shown in Slikker (2003) that relaxed sequencing games have non-empty cores.

In this section, which is based on Van Velzen and Hamers (2003) we study weak-relaxed sequencing games. These games arise from sequencing situations where exactly one agent has the power to jump. Of course, a jump is only allowed if the completion time of the agents outside the coalition does not increase. We will show that cores of weak-relaxed sequencing are non-empty by proving that these games are permutationally convex with respect to some special order.

6.5.1 Cores of weak-relaxed sequencing games

In this section we define the class of weak-relaxed sequencing games. Let (N, σ_0, α, p) be a sequencing situation and let $j \in N$. For the sake of notational simplicity we assume throughout this section that $\sigma_0(i) = i$ for each $i \in \{1, \ldots, |N|\}$. Before we define the weak-relaxed sequencing game (N, w^j), we first define relaxed sets of admissible rearrangements. In order to define these sets, we distinguish between two sets of coalitions: coalitions that include player j and coalitions that do not include player j. If $S \subseteq N$ is such that $j \notin S$, then the set of weak-relaxed admissible rearrangements, $WR^j(S)$, coincides with $A(S)$. That is, $\sigma \in WR^j(S)$ if σ satisfies (6.1). If $S \subseteq N$ is such that $j \in S$, then a reordering is called admissible if it leaves the positions of the players in $N \backslash S$ fixed, the completion times of the jobs in $N \backslash S$ do not increase and at most one jump takes place, of player j with another player in S, say player m. Formally, $\sigma \in WR^j(S)$ if

$$\sigma^{-1}(i) = \sigma_0^{-1}(i)$$

for each $i \in N \backslash S$, if

$$\sum_{k \in \{1,\ldots,|N|\}: k \leq \sigma^{-1}(i)} p_{\sigma(k)} \leq \sum_{k \in \{1,\ldots,|N|\}: k \leq \sigma_0^{-1}(i)} p_{\sigma(k)}$$

for all $i \in N \backslash S$, and if there exists an $m \in S$ such that

$$\{j \in N \backslash S : \sigma^{-1}(j) \leq \sigma^{-1}(i)\} = \{j \in N \backslash S : \sigma_0^{-1}(j) \leq \sigma_0^{-1}(j)\}$$

for all $i \in S\backslash\{j,m\}$. The corresponding *weak-relaxed sequencing game* (N, w^j) associated with j is defined by

$$w^j(S) = \sum_{i\in S} C_i(\sigma_0) - \min_{\sigma\in WR^j(S)} \sum_{i\in S} C_i(\sigma),$$

for all $S \subseteq N$. We remark that weak-relaxed sequencing games are superadditive. In fact, superadditivity of weak-relaxed sequencing games can be proved similarly to superadditivity of cps games in Lemma 6.3.2. If (N, v) is the standard sequencing game associated with (N, σ_0, α, p), then $v(S) \leq w^j(S)$ for each $S \subseteq N$ with equality if $j \notin S$ or if S is connected.

The next example illustrates weak-relaxed sequencing games. In particular it shows that weak-relaxed sequencing games need not be chain-component additive with respect to σ_0, nor convex.

Example 6.5.1 Let (N, σ_0, α, p) be a sequencing situation with $N = \{1, 2, 3\}$, $\alpha = (2, 3, 5)$ and $p = (2, 1, 1)$. Let $j = 3$. The corresponding weak-relaxed sequencing game (N, w^3) is displayed in Table 6.1.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$w^3(S)$	0	0	0	4	11	2	14

Table 6.1: The weak-relaxed sequencing game (N, w^3).

We explain $w^3(\{1,3\})$. Because $p_1 \geq p_3$ we have that $WR^3(\{1,3\}) = \{(1,2,3),(3,2,1)\}$. The cost savings for coalition $\{1,3\}$ at the processing order $(3,2,1)$ are equal to 11. Hence, $w^3(\{1,3\}) = 11$. Observe that weak-relaxed sequencing games need not be chain-component additive with respect to σ_0, since $w^3(\{1\})+w^3(\{3\}) = 0 < 11 = w^3(\{1,3\})$. Moreover, weak-relaxed sequencing games need not be convex, since $w^3(\{1,2\})+w^3(\{1,3\}) = 15 > 14 = w^3(N) + w^3(\{1\})$. $\diamond$

The upcoming theorem shows that cores of weak-relaxed sequencing games are non-empty. Specifically, it shows that the order that begins with players $j-1$ up to 1, continues with $j+1$ up to $|N|$, and ends with player j, is permutationally convex.

Theorem 6.5.1 Let (N, σ_0, α, p) be a sequencing situation and let $j \in N$. Let (N, w^j) be the corresponding weak-relaxed sequencing game. Let $\pi^j \in \Pi(N)$ be such that $\pi^j(i) = j - i$ for each $i \in \{1, \ldots, j-1\}$, $\pi^j(i) = i + 1$ for each $i \in \{j, \ldots, |N|-1\}$, and $\pi^j(|N|) = j$. Then π^j is a permutationally convex order for (N, w^j). In particular, $m^{\pi^j}(w^j) \in C(w^j)$.

Proof: We need to show

$$w^j([\pi^j(i), \pi^j] \cup S) + w^j([\pi^j(k), \pi^j]) \leq w^j([\pi^j(k), \pi^j] \cup S) + w^j([\pi^j(i), \pi^j])$$

for all $i, k \in \{1, \ldots, |N|-1\}$ with $i < k$ and $S \subseteq N \backslash [\pi^j(k), \pi^j]$ with $S \neq \emptyset$. Let $i, k \in \{0, \ldots, |N|-1\}$ with $i < k$ and $S \subseteq N \backslash [\pi^j(k), \pi^j]$ with $S \neq \emptyset$. If $i = 0$, then the inequality follows from superadditivity, so assume that $i > 0$. Observe, since $i, k < |N|$, that $j \notin [\pi^j(i), \pi^j]$ and $j \notin [\pi^j(k), \pi^j]$. Therefore, $w^j([\pi^j(i), \pi^j]) = v([\pi^j(i), \pi^j])$ and $w^j([\pi^j(k), \pi^j]) = v([\pi^j(k), \pi^j])$, where (N, v) is the standard sequencing game associated with (N, σ_0, α, p).

Now let $\sigma^{opt} \in WR^j([\pi^j(i), \pi^j] \cup S)$ be an optimal processing order for coalition $[\pi^j(i), \pi^j] \cup S$. We distinguish between two cases.

`Case 1:` $\sigma^{opt} \in A([\pi^j(i), \pi^j] \cup S)$.

Observe that

$$\begin{aligned}
& w^j([\pi^j(k), \pi^j] \cup S) + w^j([\pi^j(i), \pi^j]) \\
\geq \;& v([\pi^j(k), \pi^j] \cup S) + v([\pi^j(i), \pi^j]) \\
\geq \;& v([\pi^j(i), \pi^j] \cup S) + v([\pi^j(k), \pi^j]) \\
= \;& w^j([\pi^j(i), \pi^j] \cup S) + w^j([\pi^j(k), \pi^j]).
\end{aligned}$$

The first inequality is satisfied because $w^j([\pi^j(k), \pi^j] \cup S) \geq v([\pi^j(k), \pi^j] \cup S)$ and because $w^j([\pi^j(i), \pi^j]) = v([\pi^j(i), \pi^j])$. The second inequality follows from convexity of (N, v). The equality is satisfied because $w^j([\pi^j(i), \pi^j] \cup S) = v([\pi^j(i), \pi^j] \cup S)$ follows from the assumption that $\sigma^{opt} \in A([\pi^j(i), \pi^j] \cup S)$ and because $w^j([\pi^j(k), \pi^j]) = v([\pi^j(k), \pi^j])$.

`Case 2:` $\sigma^{opt} \notin A([\pi^j(i), \pi^j] \cup S)$.

First observe that $\sigma^{opt} \notin A([\pi^j(i), \pi^j] \cup S)$ implies that $j \in S$. Secondly, observe that our assumption implies that at σ^{opt} player j has switched position with a player, say player m, from a different component. Note that, because $[\pi^j(i), \pi^j] \cup \{j\}$ is connected, it follows that $m \notin [\pi^j(i), \pi^j]$. Hence, $m \in S$.

With loss of generality assume that $m > j$. The proof for the case $m < j$ runs similar, and is therefore omitted. Note that, because $m > j$, we obtain from the admissibility of σ^{opt} that $p_j \geq p_m$.

Note that σ^{opt} can be obtained from σ by first switching players j and m and then putting each component in decreasing order of urgency indices. So the total cost savings of $[\pi^j(i), \pi^j] \cup S$ can be decomposed into two parts as well. Namely in cost savings obtained by switching j and m, and in cost savings obtained by reordering the connected components. The cost savings obtained by switching j and m, denoted by P, are equal to

$$\begin{aligned} P = {} & \alpha_m(p_j + \ldots + p_{m-1}) - \alpha_j(p_{j+1} + \ldots + p_m) \\ & + \sum_{h \in [\pi^j(i), \pi^j] \cup S : j < h < m} (p_j - p_m)\alpha_h. \end{aligned}$$

The first term of P coincides with the cost savings obtained from moving job m up in the queue. The second term expresses the (negative) cost savings resulting from moving job j down in the queue. Finally, the completion of the jobs in between j and m decreases with an amount of $p_j - p_m$, and this explains the third term of P.

The cost savings that can be obtained by reordering the players after the switch of players j and m has taken place, can be expressed using a standard sequencing game. Specifically, let $\sigma \in Pr(N)$ be the order obtained from σ_0 by switching j and m. Formally, $\sigma(i) = i$ for each $i \in \{1, \ldots, |N|\} \backslash \{j, m\}$, $\sigma(j) = m$ and $\sigma(m) = j$. Consider the sequencing situation (N, σ, α, p) and the corresponding sequencing game (N, v_σ). Then,

$$w^j([\pi^j(i), \pi^j] \cup S) = P + v_\sigma([\pi^j(i), \pi^j] \cup S). \tag{6.79}$$

Now we provide a lower bound for the cost savings that $[\pi^j(k), \pi^j] \cup S$ can obtain. Consider the following admissible, but not necessarily optimal, reordering for $[\pi^j(k), \pi^j] \cup S$. First switch players j and m, even if these players are in the same component. Secondly, reorder the connected components using the Smith-rule. Now first observe that the cost savings obtained by the switch of j and m, denoted by Q, equal

$$\begin{aligned}Q =&\alpha_m(p_j + \ldots + p_{m-1}) - \alpha_j(p_{j+1} + \ldots + p_m)\\ &+ \sum_{h\in[\pi^j(k),\pi^j]\cup S: j<h<m} (p_j - p_m)\alpha_h.\end{aligned}$$

The cost savings obtained by rearranging the jobs after the switch of j and m equal $v_\sigma([\pi^j(k), \pi^j] \cup S)$. Hence,

$$w^j([\pi^j(k), \pi^j] \cup S) \geq Q + v_\sigma([\pi^j(k), \pi^j] \cup S). \tag{6.80}$$

From $p_j \geq p_m$ and $([\pi^j(i), \pi^j] \cup S) \subseteq ([\pi^j(k), \pi^j] \cup S)$ we conclude that $Q \geq P$. This yields

$$\begin{aligned}&w^j([\pi^j(k), \pi^j] \cup S) + w^j([\pi^j(i), \pi^j])\\ \geq\ & Q + v_\sigma([\pi^j(k), \pi^j] \cup S) + v_\sigma([\pi^j(i), \pi^j])\\ \geq\ & P + v_\sigma([\pi^j(i), \pi^j] \cup S) + v_\sigma([\pi^j(k), \pi^j])\\ =\ & w^j([\pi^j(i), \pi^j] \cup S) + w^j([\pi^j(k), \pi^j]).\end{aligned}$$

The first inequality holds by (6.80) and because $j, m \notin [\pi^j(i), \pi^j]$ imply $v_\sigma([\pi^j(i), \pi^j]) = w^j([\pi^j(i), \pi^j])$. The second inequality follows from convexity of (N, v_σ) and $Q \geq P$. The equality is due to (6.79) and because $j, m \notin [\pi^j(k), \pi^j]$ imply $v_\sigma([\pi^j(k), \pi^j]) = w^j([\pi^j(k), \pi^j])$. □

6.6 Queue allocation of indivisible objects

Many housing associations use waiting lists to allocate houses to tenants. Typically, the tenant on top of the waiting list is assigned his top choice, the tenant ordered second is assigned his top choice among the remaining houses, etc. A major reason why this mechanism is considered not very desirable is that the outcome of the procedure might not be efficient for society. In particular, by collaboration the total group of tenants might be able to achieve a higher utility.

A situation where a finite number of indivisible objects need to be allocated to the same number of individuals with respect to some queue is studied in Svensson (1994). To be more precise, Svensson (1994) discusses a situation with a finite number of indivisible objects, the same number of individuals, and an exogenously given queue. Subsequently, an allocation method is proposed and it is shown that it satisfies certain desirable properties.

This section, which is based on Hamers, Klijn, Slikker, and Van Velzen (2004), discusses a model similar to that of Svensson (1994). The main difference is that in our model we assume that the preferences of the agents over the set of objects are expressed in monetary units. This implies that the allocation proposed by Svensson (1994) might not be efficient for society. Only by collaborating will the agents be able to reach a society-efficient allocation. Because of this collaboration individual agents might not be satisfied with the final assignment of the objects. We assume that these agents are compensated by means of side-payments. Our main result is that the society-efficient assignment is supported by side-payments that guarantee stability, i.e. each coalition has an incentive to collaborate with society.

Another well-known model with indivisible objects is the housing market of Shapley and Scarf (1974). This housing market considers a finite number of agents, each initially possessing an object (house). The agents have preferences over the set of objects. It is shown that core allocations exist for this model. In Tijs, Parthasarathy, Potters, and Rajendra Prasad (1984) the model of Shapley and Scarf (1974) is adapted by assuming that the preferences of the agents can be expressed by monetary units. In this way the class of permutation games is introduced and the non-emptiness of the core is shown. Hence, our adaptation of the model of Svensson (1994) parallels the adaptation of the housing market by Tijs, Parthasarathy, Potters, and Rajendra Prasad (1984).

6.6.1 Assignment games, permutation games and extensive form games

In this section we shortly introduce some game theoretical concepts. First we recall some notions from cooperative game theory. The section ends with a brief description of extensive form games.

A *bipartite matching situation* (N, M, U) consists of two disjoint finite sets of agents N, M, and an $|N| \times |M|$-matrix U. If agents $i \in N$ and $j \in M$ collaborate they achieve a utility of $U_{ij} \in \mathbb{R}$. This matching situation was first modelled as a cooperative game in Shapley and Shubik (1972), in the following way. Let $S \subseteq N$ and $T \subseteq M$. A *matching* μ for $S \cup T$ consists of disjoint pairs in $S \times T$. Let $\mathcal{M}(S,T)$ denote the set of all matchings for coalition $S \cup T$. The *assignment game* $(N \cup M, v_A)$ is defined by $v_A(S \cup T) = \max\{\sum_{(i,j)\in\mu} U_{ij} : \mu \in \mathcal{M}(S,T)\}$ for all $S \subseteq N$ and $T \subseteq M$. That is, the worth of a coalition is obtained by maximising the sum of the utilities over the set of matchings for this coalition. A matching that maximises the sum of utilities is called optimal. It is well-known that assignment games have a non-empty core (cf. Shapley and Shubik (1972)). In particular, let μ be an optimal matching and let $x = (u, v) \in \mathbb{R}^N \times \mathbb{R}^M$. Then, $x \in C(v_A)$ if and only if $u_i + v_j = U_{ij}$ for each $(i, j) \in \mu$, $u_i + v_j \geq U_{ij}$ for each $i \in N$, $j \in M$, and $x_k \geq 0$ for each $k \in N \cup M$.

A *permutation situation* (N, M, U) consists of a finite set of agents N, a finite set of objects M, such that $|N| = |M|$, and an $|N| \times |M|$-matrix U. Each agent $i \in N$ initially possesses object $i \in M$. The utility that agent $i \in N$ receives from the consumption of object $j \in M$ is given by $U_{ij} \in \mathbb{R}$. By reallocating their initially owned objects the agents can possibly achieve a higher utility. Permutation situations can be modelled as cooperative games in the following way. A reallocation of the objects of coalition $S \subseteq N$ among the members of S can be expressed by a bijection $\pi_S : S \to O(S)$, where $O(S)$ denotes the set of objects initially owned by coalition S. Let $\Pi(S, O(S))$ denote the set of all bijections from S to $O(S)$.[9] The *permutation game* (N, v_P) is defined by $v_P(S) = \max\{\sum_{i\in S} U_{i\pi_S(i)} : \pi_S \in \Pi(S, O(S))\}$ for all $S \subseteq N$. That is, the worth of a coalition is the maximum utility it

[9] In this section we denote the set of bijections from a set A to a set B by $\Pi(A, B)$.

can achieve by reallocating their initially owned objects among its members.

Permutation games were studied first in Tijs, Parthasarathy, Potters, and Rajendra Prasad (1984). In that paper a link was established between the cores of assignment games and permutation games. It was shown that each core element of an assignment game gives rise to a core element of a related permutation game. In Quint (1996) it was shown that all core elements of a permutation game can be obtained from the core of some associated assignment game.

To conclude this section we shortly introduce *extensive form games*.[10] We first remark that we only consider extensive form games without chance nodes, but with perfect information. An extensive form game is a 4-tuple (P, T, C, u), where P is a finite set of players, T is a rooted tree with non-terminal node set V_1 and terminal node set V_2, $C : V_1 \to P$ is a control function, and $u : V_2 \to \mathbb{R}^P$ is a function expressing the utility that each player receives at each terminal node. For each $i \in P$ let $c_i \subseteq V_1$ be the set of nodes controlled by i, i.e. $c_i = \{v \in V_1 : C(v) = i\}$. A strategy of player $i \in P$ is a map $y_i : c_i \to V_1 \cup V_2$ such that $(v, y_i(v))$ is an arc in T for all $v \in c_i$. So a strategy for player i describes at each node controlled by player i the direction in which the game proceeds. The set of all strategies of player i is denoted by Σ_i. It is obvious that each strategy profile $(y_i)_{i\in P}$ leads to a unique terminal node. So each strategy profile $(y_i)_{i\in P}$ induces an outcome at the extensive form game. Hence, there exists a utility function from the set of strategy profiles $\prod_{i\in P} \Sigma_i$ to $\mathbb{R}^P$. We will denote this utility function, with slight abuse of notation, by u as well. We say that $y_i \in \Sigma_i$ is a *best reply* for player i against $y_{-i} = (y_j)_{j\in P\setminus\{i\}} \in \prod_{j\in P\setminus\{i\}} \Sigma_j$ if $u_i(y_{-i}, y_i) \geq u_i(y_{-i}, z_i)$ for all $z_i \in \Sigma_i$. In other words, a player's strategy is a best reply against some strategy profile of the other players if he cannot be strictly better off by unilaterally deviating from this strategy.

[10] For a full description of extensive form games, see for example Mas-Colell, Whinston, and Green (1995).

6.6.2 Object allocation situations and games

In this section we introduce our object allocation situation and a corresponding cooperative game.

An *object allocation situation* is a 4-tuple (N, M, U, σ_0). Here N is a finite set of agents, M is a finite set of indivisible objects, U is a non-negative $|N| \times |M|$-matrix that expresses the utility of each object for each agent, and σ_0 is an initial order on N. We assume that there are as many agents as objects, i.e. $|N| = |M|$.[11] The initial order should be interpreted as the order in which the agents may choose from the set of objects, i.e. agent $\sigma_0(1)$ has the first choice, agent $\sigma_0(2)$ the second, etc. Without loss of generality, let $\sigma_0(i) = i$ for all $i \in \{1, \ldots, |N|\}$.

Let (N, M, U, σ_0) be an object allocation situation. We will analyse this situation using cooperative game theory. At our cooperative game we define the worth $v(S)$ of a coalition $S \subseteq N$ as the maximum total utility it can guarantee itself without any help from $N \backslash S$. This utility can be determined in two stages. In the first stage, all players sequentially choose an object, respecting σ_0. In the second stage, the members of S reallocate the chosen objects among themselves to reach coalitional efficiency. Obviously, the outcome of this reallocation depends on the objects chosen by the members of S, and therefore also on the objects chosen by the members of $N \backslash S$.

In order to describe the value $v(S)$ of a coalition $S \subseteq N$, we define an (auxiliary) extensive form game $(\{S, N \backslash S\}, T, C^S, u^S)$ with player set $\{S, N \backslash S\}$. We first describe the rooted tree T. Let $k \in \{1, \ldots, |M|\}$. The set of injective maps from $\{1, \ldots, k\}$ to M is denoted by $\mathcal{S}_k$. A map $\pi \in \mathcal{S}_k$ is interpreted as a situation where object $\pi(i)$ is chosen by agent i for each $i \in \{1, \ldots, k\}$. Similarly, we define $\mathcal{S}_0$ as the situation where none of the objects is chosen yet. Let T be the rooted tree with node set $\bigcup_{k=0}^{|M|} \mathcal{S}_k$ and root $\mathcal{S}_0$. There is an arc between $\pi \in \mathcal{S}_k$ and $\tau \in \mathcal{S}_{k+1}$ with $k \in \{0, \ldots, |M| - 1\}$, if and only if $\pi(i) = \tau(i)$ for all $i \in \{1, \ldots, k\}$. That is, there is an arc between π and τ if π can be extended to τ by assigning object $\tau(k + 1)$ to player $k + 1$. So, $V_1 = \bigcup_{k=0}^{|M|-1} \mathcal{S}_k$ and $V_2 = \mathcal{S}_{|M|}$ are the sets of non-terminal and

[11] The situation where $|M| < |N|$ is captured by introducing worthless null objects.

terminal nodes, respectively.

We define the control function $C^S : \bigcup_{k=0}^{|M|-1} \mathcal{S}_k \to \{S, N\backslash S\}$ as follows. Let $\pi \in \mathcal{S}_k$ for some $k \in \{0, \ldots, |M|-1\}$. Then we define $C^S(\pi) = S$ if and only if $k+1 \in S$. So coalition S controls the nodes at which one of its members is to choose an object. Let Σ_S and $\Sigma_{N\backslash S}$ be the set of all possible strategies of players S and $N\backslash S$, respectively.

Finally, we describe the utility function $u^S : \Sigma_S \times \Sigma_{N\backslash S} \to \mathbb{R}^{\{S, N\backslash S\}}$. Let $y = (y_S, y_{N\backslash S}) \in \Sigma_S \times \Sigma_{N\backslash S}$. Let $\tau \in \mathcal{S}_m$ be the terminal node reached by strategy profile y, and let $H_S(\tau) = \{\tau(i) : i \in S\}$ be the corresponding set of objects obtained by S. Now define $u^S_S(y) = \max\{\sum_{i\in S} U_{i\pi(i)} : \pi \in \Pi(S, H_S(\tau))\}$, and $u^S_{N\backslash S}(y) = -u^S_S(y)$. So, the payoff of S at terminal node $\tau \in \mathcal{S}_m$ is the maximum utility S obtains after reallocating the initially chosen objects and the payoff for $N\backslash S$ is just the opposite of the payoff of S. Hence, $N\backslash S$ maximises its payoff at the extensive form game by minimising the payoff of S.

Now we define the *object allocation game* (N, v) by

$$v(S) = \max_{y_S \in \Sigma_S} \min_{y_{N\backslash S} \in \Sigma_{N\backslash S}} u^S_S(y) \text{ for all } S \subseteq N.$$

Note that $v(S)$ is precisely the maximum utility coalition S can guarantee itself without any help from $N\backslash S$. Also, notice that $v(N) = v_A(N \cup M)$, where $(N \cup M, v_A)$ is the assignment game corresponding to the bipartite matching situation (N, M, U).

We illustrate the object allocation game in the following example.

Example 6.6.1 Let $N = \{1, 2, 3\}$, $M = \{A, B, C\}$ and $U = \begin{pmatrix} 3 & 6 & 2 \\ 4 & 5 & 3 \\ 5 & 3 & 0 \end{pmatrix}$.

The object allocation game (N, v) is given by

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	6	4	0	10	7	6	14

To see for instance why $v(\{1,3\}) = 7$ consider the extensive form game $(\{\{1,3\},\{2\}\}, T, C^{\{1,3\}}, u^{\{1,3\}})$ which is depicted in Figure 6.2. If coalition $\{1,3\}$ chooses object A as a first choice, then the utility it will achieve is equal to 7, since player $\{2\}$ will choose object B in order to maximise its own payoff at the extensive form game. If coalition $\{1,3\}$ chooses object B or C first, then coalition $\{2\}$ will obviously maximise its payoff at the extensive form game by choosing object A. This leads to a utility of 6 for coalition $\{1,3\}$. Hence, coalition $\{1,3\}$ can guarantee itself a payoff of 7 by first choosing object A. We conclude that $v(\{1,3\}) = 7$. $\diamond$

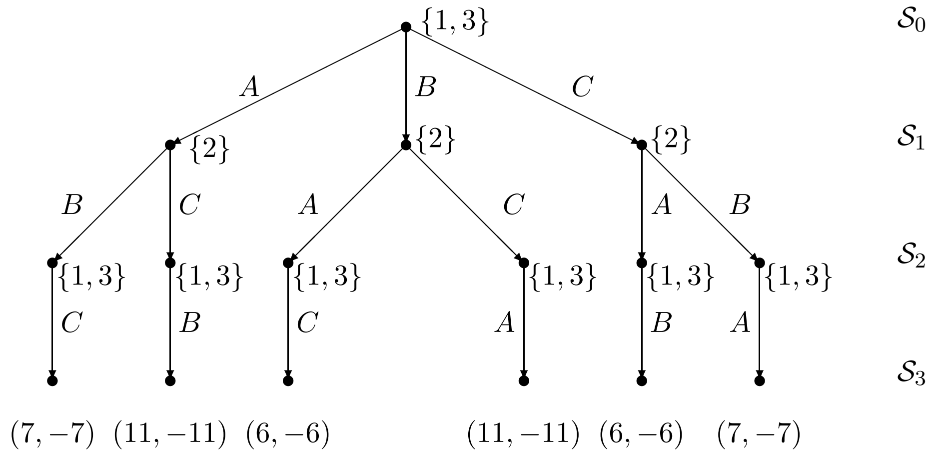

Figure 6.2: The extensive form game $(\{\{1,3\},\{2\}\}, T, C^{\{1,3\}}, u^{\{1,3\}})$.

6.6.3 Cores of object allocation games

In this section we show that the core of object allocation games is non-empty. In fact, we provide a method to obtain core elements by using core elements from a related assignment game. Furthermore, we show that for a special class of utility profiles the object allocation game coincides with a corresponding permutation game.

The following theorem shows the non-emptiness of the core of object allocation games.

Theorem 6.6.1 Let (N, M, U, σ_0) be an object allocation problem and let (N, v) be its corresponding game. Let (N, M, U) be the corresponding bipartite matching problem and let $(N \cup M, v_A)$ be the corresponding assignment game. Let $(u, w) \in C(v_A)$ and let $\tau : \{1, \ldots, |M|\} \to M$ be a bijection such that $w_{\tau(1)} \geq \ldots \geq w_{\tau(|M|)}$. Define $x_i = u_i + w_{\tau(i)}$ for all $i \in N$. Then, $x \in C(v)$.

Proof: By definition of x, $\sum_{i \in N} x_i = v_A(N \cup M)$. Since $v_A(N \cup M) = v(N)$, $\sum_{i \in N} x_i = v(N)$. It remains to show that $\sum_{i \in S} x_i \geq v(S)$ for each $S \subseteq N$. Let $S \subseteq N$ and consider the extensive form game $(\{S, N \backslash S\}, T, C^S,\ u^S)$. Consider the following (possibly non-optimal) strategy $z_{N \backslash S} \in \Sigma_{N \backslash S}$ for player $N \backslash S$: "always pick the object with highest w_i that is still available." More precisely, let $z_{N \backslash S} \in \Sigma_{N \backslash S}$ be such that $z_{N \backslash S}(\sigma) = \pi$ for each $\sigma \in \mathcal{S}_k$, $k + 1 \in N \backslash S$, and $\pi \in \mathcal{S}_{k+1}$ with $w_{\pi(k+1)} \geq w_j$ for all $j \in M \backslash \{\sigma(1), \ldots, \sigma(k)\}$.

Now if player S would use a similar strategy in the strategic form game as player $N \backslash S$, i.e. also "always pick the highest w_i that is still available," then player S would acquire $\{\tau(i) : i \in S\}$ as its set of objects. If player S uses a different strategy, then, given player $N \backslash S$'s strategy $z_{N \backslash S}$, it would obtain a set of objects A with lower w_i-values. Formally,

$$\sum_{a \in A} w_a \leq \sum_{i \in S} w_{\tau(i)}. \tag{6.81}$$

In particular, let player S play a best reply against strategy $z_{N \backslash S}$. Let A^* be the set of objects acquired by S. Let $\pi : S \to A^*$ be the optimal reallocation of the obtained objects. From (6.81) it follows that

$$\sum_{i \in S} w_{\pi(i)} = \sum_{a \in A^*} w_a \leq \sum_{i \in S} w_{\tau(i)}. \tag{6.82}$$

Hence,

$$\begin{aligned} \sum_{i \in S} x_i = \sum_{i \in S} u_i + \sum_{i \in S} w_{\tau(i)} &\geq \sum_{i \in S} u_i + \sum_{i \in S} w_{\pi(i)} \\ &\geq v_A(S \cup \{\pi(i) : i \in S\}) = \sum_{i \in S} U_{i\pi(i)}. \end{aligned} \tag{6.83}$$

The first inequality is due to (6.82). The second inequality is satisfied because $(u, w) \in C(v_A)$. The last equality is satisfied since the matching

$\{(i,\pi(i)) : i \in S\}$ is an optimal reallocation, and hence optimal for coalition $S \cup \{\pi(i) : i \in S\}$ at the assignment game (N, v_A).

From the definition of the game (N, v) it follows that

$$\sum_{i \in S} U_{i\pi(i)} = \max_{y_S \in \Sigma_S} u_S^S(y_S, z_{N\setminus S}) \geq \max_{y_S \in \Sigma_S} \min_{y_{N\setminus S} \in \Sigma_{N\setminus S}} u_S^S(y) = v(S). \quad (6.84)$$

Now the theorem follows immediately from (6.83) and (6.84). □

The next example illustrates Theorem 6.6.1. Moreover, it shows that in general not all core elements of the object allocation game can be obtained via the technique of Theorem 6.6.1.

Example 6.6.2 Let (N, M, U, σ_0) be the object allocation situation from Example 6.6.1, and (N, v) the corresponding game. Consider the corresponding bipartite matching situation (N, M, U) and assignment game $(N \cup M, v_A)$. Note that $(u, w) \in C(v_A)$ with $u = (2, 3, 3)$ and $w = (w_A, w_B, w_C) = (2, 4, 0)$. Clearly $w_B \geq w_A \geq w_C$. Now let $x_1 = u_1 + w_B = 6$, $x_2 = u_2 + w_A = 5$, and $x_3 = u_3 + w_C = 3$. From Theorem 6.6.1 it follows that $x = (6, 5, 3) \in C(v)$.

We will now show that not each element of $C(v)$ is achievable by the method of Theorem 6.6.1. Consider $y = (8, 4, 2) \in C(v)$. Suppose that $(u', w') \in C(v_A)$ is such that $u'_1 + w'_{\tau(1)} = 8$, $u'_2 + w'_{\tau(2)} = 4$, and $u'_3 + w'_{\tau(3)} = 2$ where $\tau : \{1, 2, 3\} \to M$ is a bijection with $w'_{\tau(1)} \geq w'_{\tau(2)} \geq w'_{\tau(3)}$.

First note that (N, M, U) has a unique optimal matching $\mu = \{(1, B), (2, C), (3, A)\}$. So, since $(u', w') \in C(v_A)$, it holds that $u'_1 + w'_B = U_{1B} = 6$, $u'_2 + w'_C = U_{2C} = 3$, and $u'_3 + w'_A = U_{3A} = 5$. Since $u'_2 + w'_{\tau(2)} = 4$ and $u'_2 + w'_C = 3$, it follows that $w'_{\tau(2)} > w'_C$. So, $w'_{\tau(1)} \geq w'_{\tau(2)} > w'_C$. Hence, $\tau(3) = C$. Because $u'_1 + w'_{\tau(1)} = 8$ it follows that $w'_{\tau(1)} > w'_B$, and thus that $\tau(1) \neq B$. We conclude that $w'_A \geq w'_B \geq w'_C$. Hence, $u'_2 + w'_B = u'_2 + w'_{\tau(2)} = 4 < 5 = v_A(\{2, B\})$ contradicting $(u', w') \in C(v_A)$. ◇

Our second result deals with a special case of object allocation situations. Let (N, M, U, σ_0) be an object allocation situation where all agents prefer the first object over the second, the second object over the third, etc. Then, the object allocation game coincides with a corresponding permutation game.

Proposition 6.6.1 Let (N, M, U, σ_0) be an object allocation situation with $U_{j1} \geq \ldots \geq U_{j|M|}$ for all $j \in N$ and let (N, v) be its corresponding object allocation game. Let (N, M, U) be the corresponding permutation situation and (N, v_P) its corresponding game. Then, the games (N, v) and (N, v_P) coincide.

Proof: We show that for all $S \subseteq N$ it holds that $v(S) = v_P(S)$. Let $S \subseteq N$ and consider the extensive form game $(\{S, N\backslash S\}, T, C^S, u^S)$. First we show, by giving a strategy for player S, that at the extensive form game player S can obtain a payoff of at least $v_P(S)$. This implies $v(S) \geq v_P(S)$.

Consider the following strategy $z_S \in \Sigma_S$ for player S at the extensive form game: "always pick the remaining object with lowest index number," i.e. pick the remaining object with highest utility. In other words, z_S is such that $z_S(\sigma) = \tau$ for each $\sigma \in \mathcal{S}_k$, $k+1 \in S$, and $\tau \in \mathcal{S}_{k+1}$ with $\tau(k+1) = \min\{j : j \in M\backslash\{\sigma(1), \ldots, \sigma(k)\}\}$. Let the best reply of player $N\backslash S$ against this strategy of S result in a set of objects $A = \{a_1, \ldots, a_{|S|}\} \subseteq M$ for S. We assume, without loss of generality, that the elements of A are ordered $a_1 < a_2 < \ldots < a_{|S|}$.[12]

Denote the set of objects initially owned by S in the permutation situation (N, M, U) by $O(S) = \{b_1, \ldots, b_{|S|}\}$. We assume, without loss of generality, that this set is ordered $b_1 < b_2 < \ldots < b_{|S|}$. Note that by definition of strategy z_S player S will obtain a better set of objects at the extensive form game in the sense that $a_j \leq b_j$ for all $j \in \{1, \ldots, |S|\}$. Now let $\pi^* : S \to B$ be the optimal reallocation of the objects in B among the members of S, i.e. $\sum_{i \in S} U_{i\pi^*(i)} = \max\{\sum_{i \in S} U_{i\pi(i)} : \pi \in \Pi(S, B)\}$. Furthermore, define $\bar{\pi} : S \to A$ by $\bar{\pi}(i) = a_j$ if and only if $\pi^*(i) = b_j$. In other words, assign the j-th object of A to player i if and only if it is optimal to assign the j-th object of B to i. Now

$$v_P(S) \quad = \quad \max\{\sum_{i \in S} U_{i\pi(i)} : \pi \in \Pi(S, B)\}$$

[12] Recall that $M = \{1, \ldots, m\}$ and that $a_i < a_k$ implies that $U_{ji} \geq U_{jk}$ for all $j \in N$.

$$
\begin{aligned}
&= \sum_{i \in S} U_{i\pi^*(i)} \\
&\leq \sum_{i \in S} U_{i\bar{\pi}(i)} \\
&\leq \max\{\sum_{i \in S} U_{i\pi(i)} : \pi \in \Pi(S, A)\} \\
&\leq v(S).
\end{aligned}
$$

The first inequality holds because $\pi^*(i) \geq \bar{\pi}(i)$ for all $i \in S$. The second inequality is satisfied because $\bar{\pi}$ might be non-optimal. The last inequality is satisfied since the strategy of S might be non-optimal.

Finally, we prove the inequality $v(S) \leq v_P(S)$ by considering the following strategy for $N \backslash S$: "always pick the remaining object with lowest index number." It is obvious that if $N \backslash S$ uses this strategy, then S cannot do better than to obtain the set of objects B. By reallocation of the objects in B player S obtains a maximal total utility of $v_P(S)$. Therefore, $v(S) \leq v_P(S)$. $\square$

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Subject index

Samenwerking in netwerken en volgordeproblemen

Samenvatting

Het onderwerp van deze verhandeling is coöperatieve speltheorie. Deze tak van wetenschap houdt zich voornamelijk bezig met het wiskundig beschrijven en analyseren van samenwerkingsverbanden binnen een economische context. Met name de totstandkoming en de verdeling van synergievoordelen ontstaan door samenwerking zijn onderwerp van studie.

Binnen de coöperatieve speltheorie wordt een wiskundig model van synergievoordelen ontsproten aan een samenwerkingsverband aangeduid met de term "spel". De betrokken economische agenten worden heel toepasselijk "spelers" genoemd. Het bekendst zijn spelen met overdraagbaar nut. Een spel met overdraagbaar nut beschrijft voor iedere groep spelers de monetaire waarde van de synergievoordelen die deze groep kan genereren door middel van samenwerking. Deze waardes kunnen vervolgens worden gebruikt om te bepalen welke samenwerkingsverbanden zullen ontstaan, en hoe de synergievoordelen zullen worden verdeeld. Het moge duidelijk zijn dat deze twee vraagstukken gerelateerd zijn. De uiteindelijke samenwerkingsverbanden zullen afhangen van de verdeling van de synergievoordelen, en de verdeling hangt weer af van de ontstane samenwerkingsverbanden. Eén van de voornaamste doelstellingen binnen de coöperatieve speltheorie is het "eerlijk" verdelen van synergievoordelen. Met dit doel is er een scala aan oplossingsconcepten ontwikkeld. Een oplossingsconcept is grof gezegd een afbeelding die aan ieder coöperatief spel één of meerdere verdelingen

toekent.

Dit proefschrift behandelt verscheidene aspecten van coöperatieve speltheorie. Ten eerste poogt het relaties tussen eigenschappen en oplossingsconcepten van spelen te onderzoeken. Ten tweede worden verscheidene samenwerkingsverbanden gemodelleerd als spelen, en worden deze spelen vervolgens onderzocht.

Deze verhandeling begint met een inleidend hoofdstuk, Hoofdstuk 1. Dit hoofdstuk geeft een korte introductie tot de coöperatieve speltheorie, alsmede een bespreking van de meest elementaire wiskundige begrippen en notaties die dit proefschrift sieren. Het hoofdstuk eindigt met een vooruitblik naar de volgende hoofdstukken.

Hoofdstuk 2 is geheel en al gewijd aan marginale vectoren. Dit zijn oplossingsconcepten die gerelateerd zijn aan volgordes op de spelersverzameling. Binnen de speltheorie is het welbekend dat er een relatie bestaat tussen marginalen en convexiteit. In Hoofdstuk 2 wordt deze relatie verder uitgeplozen. Er wordt gepoogd verzamelingen marginalen te construeren met de volgende eigenschap: als de marginalen in deze verzameling zich in de kern van een spel bevinden, dan is dit spel noodzakelijkerwijs convex. Het hoofdresultaat van het hoofdstuk is de bepaling van de minimale cardinaliteit van zulke verzamelingen, en een constructieve methode om tot zulke minimale verzamelingen te komen.

Het onderwerp van Hoofdstuk 3 is boom-component additieve spelen. Dit zijn superadditieve spelen waarbij de mogelijke samenwerkingsverbanden van de spelers beperkt worden door een exogeen gegeven boom, d. i. alleen samenhangende groepen spelers worden in staat geacht synergievoordelen te genereren. Het hoofdstuk richt zich met name op eigenschappen als stabiliteit van de kern, exactheid, uitbreidbaarheid en grootheid van de kern. Voor het speciale geval van keten-component additieve spelen wordt stabiliteit van de kern zelfs gekarakteriseerd.

Het daaropvolgende hoofdstuk bespreekt kostenverdelingen bij de plaatsing van faciliteiten. Hierbij wordt gebruik gemaakt van het graaftheoretische concept "dominerende verzameling". Er worden drie verschillende spelen geïntroduceerd die het kostenverdelingsprobleem modelleren. Deze spelen verschillen door de mogelijkheden die groepen spelers hebben om faciliteiten te plaatsen. Er worden relaties tussen de spelen onderzocht, alsmede eigenschappen als niet-leegheid van de kern en convexiteit.

Hoofdstuk 5 houdt zich bezig met de verdeling van onderhoudskosten van netwerken. Dit is een bekend kostenverdelingsprobleem binnen de coöperatieve speltheorie. Echter, in de literatuur wordt de aanname gemaakt dat iedere speler zich op precies één plek in het netwerk bevindt. In Hoofdstuk 5 wordt met behulp van een voorbeeld aangetoond dat deze aanname niet altijd realistisch is. Vervolgens beschouwt het hoofdstuk kostenverdelingsproblemen waarbij de aanname niet geldt. Voor de bijbehorende spelen worden niet-leegheid van de core aangetoond, alsmede verscheidene oplossingsconcepten bestudeerd.

Het laatste hoofdstuk is geheel gewijd aan machinevolgordeproblemen. Bij een machinevolgordeprobleem is er een machine die verscheidene taken moet uitvoeren die beheerd worden door verschillende spelers. De machine kan slechts één taak tegelijk afhandelen, dus staan de spelers netjes in de rij te wachten alvorens hun taak afgehandeld wordt. Aangezien niet alle taken even urgent zijn, kunnen er kosten bespaard worden door spelers van plaats te laten wisselen. De vraag is nu natuurlijk hoe deze kostenbesparingen verdeeld dienen te worden over de spelers. In Hoofdstuk 6 worden verscheidene machinevolgordeproblemen bestudeerd. Ten eerste wordt een model beschouwd waarin de spelers niet alleen de mogelijkheid hebben van plaats te wisselen, maar tevens de mogelijkheid hebben de afhandelingstijd van hun taak te reduceren, tegen een verhoogd tarief. Voor dit model wordt de kern bestudeerd, alsmede convexiteit in enkele speciale gevallen. Ten tweede wordt een model geïntroduceerd waarin voorrangsrelaties een rol spelen. Er wordt aangetoond dat als de voorrangsrelatie bestaat uit een serie parallelle ketens, en de initiële volgorde een aaneenschakeling van deze ketens is, dat dan het bijbehorende machinevolgordespel convex is. Vervolgens volgt er een korte beschouwing over machinevolgordespelen met een ruimere verzameling toegelaten herrangschikkingen. Voor deze spelen wordt niet-leegheid van de kern aangetoond. Het laatste gedeelte van het hoofdstuk, en proefschrift, beschouwt de verdeling van een eindig aantal ondeelbare objecten over een zelfde aantal spelers. Er wordt aangetoond dat samenwerking ondersteund wordt door stabiele zijbetalingen.